# Algorithmic Game Theory and Applications

## Lecture 8: Computing solutions for General Strategic Games: Part 1: dominance and iterated strategy elimination

Kousha Etessami

# a partial-order on strategies: dominance

Let's again consider general finite strategic games.

**Definition** For $x_i, x_i' \in X_i$, we say $x_i$ **dominates** $x_i'$, denoted $x_i \succeq x_i'$, if for all $x_{-i} \in X_{-i}$,

$$U_i(x_{-i}; x_i) \geq U_i(x_{-i}; x_i')$$

We say $x_i$ **strictly dominates** $x_i'$, denoted $x_i \succ x_i'$, if for all $x_{-i} \in X_{-i}$

$$U_i(x_{-i}; x_i) > U_i(x_{-i}; x_i')$$

**Proposition** $x_i$ dominates $x_i'$ if and only if for all pure "counter profiles" $\pi_{-i} \in X_{-i}$

$$\pi_{-i} = (\pi_{1,j_1}, \ldots, \texttt{empty}, \ldots, \pi_{n,j_n}),$$

$$U_i(\pi_{-i}; x_i) \geq U_i(\pi_{-i}; x_i').$$

Likewise, $x_i$ strictly dominates $x_i'$ iff for all $\pi_{-i}$

$$U_i(\pi_{-i}; x_i) > U_i(\pi_{-i}; x_i')$$

**Proof** Another easy "weighted average" argument. $\square$

# obviously good strategies: dominant strategies

**Definition** A mixed strategy $x_i \in X_i$ is **dominant** if for all $x'_i \in X_i$, $x_i \succeq x'_i$. $x_i$ is **strictly dominant** if for all $x'_i \in X_i$ such that $x'_i \neq x_i$, $x_i \succ x'_i$.

**Definition** For a mixed strategy $x_i$, its **support**, $support(x_i)$, is the set of pure strategies $\pi_{i,j}$ such that $x_i(j) > 0$.

**Proposition** Every dominant strategy $x_i$ is in fact a "weighted average" of pure dominant strategies. I.e., each $\pi_{i,j} \in support(x_i)$ is also dominant.

Moreover, only a pure strategy can be strictly dominant.

**Proof** Again, easy "weighted average" argument:

$$U_i(x_{-i}; x_i) = \sum_{j=1}^{m_i} x_i(j) * U_i(x_{-i}; \pi_{i,j}).$$

If $x_i$ is dominant, then for any $x_{-i}$ $U_i(x_{-i}; x_i) \geq U_i(x_{-i}; \pi_{i,j})$, for all $j$. But then if $x_i(j) > 0$, $U_i(x_{-i}; x_i) = U_i(x_{-i}; \pi_{i,j})$.

If $x_i$ is strictly dominant, it must clearly be equal to the unique pure strategy in its support. $\square$

# So, easy algorithm to find dominant strategies

- For each player $i$ and each pure strategy $s_j \in S_i$,
  - Check if, for all pure combinations $s \in S = S_1 \times \ldots S_n$, $u_i(s_{-i}; s_j) \geq u_i(s)$.
  - If this is so for all $s$, output "$s_j$ is a dominant strategy for player $i$".
- If no such pure strategy found, then there are no dominant strategies.

Same easy algorithm for a strictly dominant strategy.
But there may be no dominant strategies. . . .

# obviously bad: strictly dominated strategies

**Definition** We say a strategy $x_i \in X_i$ is **strictly dominated** if there exists another strategy $x_i'$ such that $x_i' \succ x_i$. We say $x_i$ is **weakly dominated** if there exists $x_i'$ such that $x_i' \succeq x_i$ and for some $x_{-i} \in X_{-i}$, $U_i(x_{-i}; x_i') > U_i(x_{-i}; x_i)$.

Clearly, strictly dominated strategies are "*bad*": "rational" players would be stupid to play them.

Weakly dominated strategies aren't necessarily as "bad". It depends on what you think others will play. In particular, there can be Nash Equilibria where everybody is playing a weakly dominated strategy:

$$\begin{bmatrix} (0,0) & (0,0) \\ (0,0) & (1,1) \end{bmatrix}$$

**Question** How can we compute whether a strategy is (strictly) dominated?

**Example** Consider the following table, showing only Player 1's payoffs: Is the last row strictly dominated?

$$\begin{bmatrix} 30 & 0 & 0 \\ 0 & 30 & 0 \\ 0 & 0 & 30 \\ 5 & 5 & 5 \end{bmatrix}$$

# finding strictly dominated strategies via LP

**Goal:** Determine if $x_i \in X_i$ is (strictly) dominated.

To do this, we can use an LP with strict inequalities.
For each pure "counter profile" $\pi_{-i}$, we add a constraint $C_{\pi_{-i}}(x'_i(1), \ldots, x'_i(n))$, given by:

$$U_i(\pi_{-i}; x'_i) > U_i(\pi_{-i}; x_i)$$

Note that this is a linear constraint: the right hand side is a constant we can compute, and the left hand side is linear in the variables $x'_i(1), \ldots, x'_i(n)$.
We also add the constraints $x'_i(1) + \ldots + x'_i(n) = 1$, and $x'_i(j) \geq 0$, for $j = 1, \ldots, n$.
$x_i$ is strictly dominated iff this "strict LP" is feasible.

**Question:** But how do we cope with strict inequalities?

# Coping with strict inequalities when checking feasibility of LP constraints

- Introduce a new variable $y \geq 0$, to be `Maximized`, and change constraints to:

$$U_i(\pi_{-i}; x'_i) \geq U_i(\pi_i; x_i) + y$$

- Then $x_i$ is strictly dominated if and only if the new LP (with objective "`Maximize y`") is feasible and the optimal value for $y$ is $> 0$ (or unbounded, but in this particular example that can't happen).
- Observe: Any optimal solution $x'_i$ to this revised LP is itself not strictly dominated.
- **Note:** This provides a general recipe for converting the problem of checking *feasibility* of any set of linear constraints *including strict inequalities*, to a new LP optimization problem, *without* strict inequalities.

# common knowledge and strategy elimination

- Recall the games "Guess Half the Average", and "Give a (matched) dollar to the other player".

- How do we reason about such games? Suppose I "know" that all players are "*rational*" (i.e., aim to optimize their own expected payoff). Then I might conclude: "Jane will never play a strictly dominated (SD) strategy. So I can eliminate her SD strategies from consideration." But by eliminating her SDSs, some of my strategies may become SD'ed! Deepening the reasoning, suppose

  *"I know that she knows that I know that ......"*.

- **Definition (somewhat informal):** A fact $P$ is "**common knowledge**" among all $n$ players if:
    - For every player $i$, "Player $i$ knows $P$": call this fact $P_{\langle i \rangle}$.
    - And, inductively, for $k \geq 1$, for all players $i$, and all sequences $s = i_1 \ldots i_k \in \{1, \ldots, n\}^k$, "Player i knows $P_{\langle s \rangle}$": call this fact $P_{\langle i\, s \rangle}$.

  (To be more formal, we would have to delve deeper into "*logics of knowledge*". Outside the scope of this class. See, e.g., the book [Fagin-Halpern-Moses-Vardi'95].)

- **RKN hypothesis:** every player's "rationality" is common knowledge among all players.

# iterated SDS elimination algorithm

Assuming the RKN hypothesis, we can safely conduct the following strategy elimination algorithm:

- **While** (some pure strategy $\pi_{i,j}$ is SD'ed)
  eliminate $\pi_{i,j}$ from the game,
  obtaining a new residual game;

Sometimes, a player's strategy may be uniquely determined by the end of elimination, but certainly not always:

$$\begin{bmatrix} (0,7) & (2,5) & (7,0) & (0,1) \\ (5,2) & (3,3) & (5,2) & (0,1) \\ (7,0) & (2,5) & (0,7) & (0,1) \\ (0,0) & (0,-2) & (0,0) & (10,-1) \end{bmatrix}$$

**Note:** we iteratively eliminate only pure SDSs. There may in fact remain mixed SDSs. Before playing any mixed strategy in the residual game we should make sure it is not SD'ed (by, e.g., checking it is an optimal solution of appropriate LP).

# remarks

- There is a more general notion of **rationalizability** ([Bernheim'84,Pierce84]), which says:
  - A rational player $i$ should never play a strategy $x_i$ which is "never a best response" to any counter strategy $x_{-i}$ (see below).
  - Assuming rationality is common knowledge, we should also iteratively eliminate all strategies that are "never a best response".
- It turns out, for 2-player games, this elimination yields exactly the same residual game as iterated SDS elimination. So the same algorithm applies.
- For $> 2$ players, things get more complicated: this equivalence doesn't hold unless we adopt a different notion of "never a best response" (with respect to any "<u>*belief*</u>" of player $i$ about other players' strategies, ...... we will not consider this further.)

# weakly vs. strictly dominated strategies

- Note: We did not eliminate weakly dominated strategies.
- In fact, the residual game obtained from iterated WDS elimination depends on the order of elimination:

$$\begin{bmatrix} (5,1) & (4,0) \\ (6,0) & (3,1) \\ (6,4) & (4,4) \end{bmatrix}$$

- This problem does not arise for strictly dominated strategies:
  **Proposition** Iterated elimination of strictly dominated strategies produces the same final residual game regardless of the order in which strategies are eliminated.
  **Proof** If a pure strategy is strictly dominated, it will remain strictly dominated even after another strictly dominated pure strategy is removed. $\square$

# Computing Nash Equilibria: a first clue

Recall "Useful corollary for NEs", from Lecture 3:

> *If $x^*$ is an NE and $x_i^*(j) > 0$ then*
> $U_i(x^*_{-i}; \pi_{i,j}) = U_i(x^*)$.

Using this, we can fully characterize Nash Equilibria:

**Proposition** In an $n$-player game, a profile $x^*$ is a Nash Equilibrium if and only if there exist $w_1, \ldots, w_n \in \mathbb{R}$, such that the following hold:

1. For all players $i$, and every $\pi_{i,j} \in \mathit{support}(x_i^*)$,
   $U_i(x^*_{-i}; \pi_{i,j}) = w_i$, and
2. For all players $i$, and every $\pi_{i,j} \notin \mathit{support}(x_i^*)$,
   $U_i(x^*_{-i}; \pi_{i,j}) \leq w_i$.

Note: such $w_i$'s necessarily satisfy $w_i = U_i(x^*)$.

**Proof** Easy from what we already know. □

**Food for thought:** Can you use this to find a NE?