Algorithmic Game Theory and Applications

Lecture 1: What is game theory?

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Basic course information

- **Lecturer**: Kousha Etessami;
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- **Lecture times: Monday**&**Thursday**, 11:10-12:00; All lectures are in person in Appleton Tower Lecture Theatre 2 (lectures will be recorded, and the recordings will be accessible via the LEARN page for the course).
  Tutorials: weekly, starting in week 3, based on weekly tutorial sheets. There will be two tutorial groups; current plan is for tutorials on Tuesdays 11:10-12:00 and Wednesdays 11:10 – 12:00 (both to be confirmed).
- **Assessments:** two written courseworks (each counts as 10% of overall mark), and one final exam (counts as 80%).
- **Course's Main Web Page** (with lecture notes/reading list, courseworks, tutorial sheets, etc):
  `http://www.inf.ed.ac.uk/teaching/courses/agta/`

- **No required textbook.** Course based on lecture notes + assigned readings. *Some useful reference textbooks:*

M. Maschler, E. Solan, & S. Zamir, *Game Theory*, 2013.

M. Osborne and A. Rubinstein, *A Course in Game Theory*, 1994.

R. Myerson, *Game Theory: Analysis of conflict*, 1991.

A. Mas-Colell, M. Whinston, and J. Green, Microeconomic Theory, 1995.

N. Nisan, T. Roughgarden, E. Tardos, and V. Vazirani (editors), *Algorithmic Game Theory*, 2007. *(Available for free online.)*

T. Roughgarden *Twenty Lectures on Algorithmic Game Theory*, Cambridge U. Press, 2016. (Available online from Library.)

Y. Shoham & K. Leyton-Brown, *Multiagent systems: algorithmic, game-theoretic, and logical foundations*, 2009. (Available from Library online.)

V. Chvátal, *Linear Programming*, 1980.

What is Game Theory?

A general and vague definition:

> *"Game Theory is the formal study of interaction between 'goal-oriented' 'agents' (or 'players'), and the strategic scenarios that arise in such settings."*

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What is *Algorithmic Game Theory?*

> *"Concerned with the computational questions that arise in game theory, and that enlighten game theory. In particular, questions about finding efficient algorithms to 'solve' games."*

These vague sentences are best illustrated by looking at examples.

A simple 2-person game: Rock-Paper-Scissors

		Player II		
		Rock	Paper	Scissors
Player I	Rock	0, 0	−1, 1	1, −1
	Paper	1, −1	0, 0	−1, 1
	Scissors	−1, 1	1, −1	0, 0

- This is a “zero-sum” game: whatever Player I wins, Player II loses, and vice versa.
- What is an “optimal strategy” in this game?
- How do we compute such “optimal strategies” for 2-person zero-sum games?

A non-zero-sum 2-person game: Prisoner's Dilemma

		Player II	
		Cooperate	Defect
Player I	Cooperate	2 (I), 2 (II)	0 (I), 3 (II)
	Defect	3 (I), 0 (II)	1 (I), 1 (II)

- For both players Defection is a *"Dominant Strategy"* (regardless what the other player does, you're better off Defecting).
- But if they both Cooperate, they would both be better off.
- Game theorists/Economists worry about this kind of situation as a real problem for society.
- Often, there are no "dominant strategies". What does it mean to "solve" such games?

Nash Equilibria

- A Nash Equilibrium (NE) is a pair (n-tuple) of strategies for the 2 players (n players) such that no player can benefit by unilaterally deviating from its strategy.

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- **Nash's Theorem:** Every (finite) game has a *mixed* (i.e., randomized) Nash equilibrium.
- **Example 1:** The pair of dominant strategies (Defect, Defect) is a pure NE in the Prisoner's Dilemma game. (In fact, it is the only NE.)
  In general, there may be many NE, none of which are pure.
- **Example 2:** In Rock-Paper-Scissors, the pair of *mixed* strategies: ((R=1/3, P=1/3,S=1/3), (R=1/3, P=1/3,S=1/3)) is a Nash Equilibrium. (And, we will learn, it is also a *minimax* solution to this zero-sum game. The *"minimax value"* is 0, as it must be because the game is *"symmetric"*.)
- **Question:** How do we compute a Nash Equilibrium for a given game?

Multiple equilibria

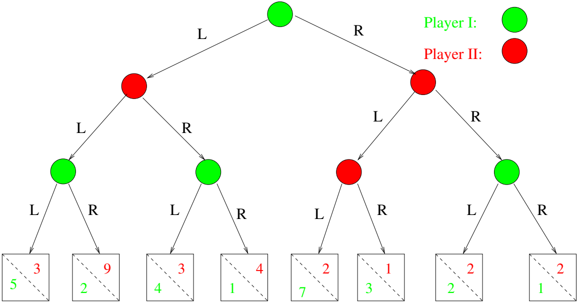
- Many games have > 1 NE. **Example:** A "Coordination Game":

		Player II	
		A	B
Player I	A	2, 2	0, 0
	B	0, 0	1, 1

- There are two **pure** Nash Equilibria: (A, A) and (B, B).
  Are there any other NEs?
  Yes, there's one other *mixed* (randomized) NE.

Games in "Extensive Form"

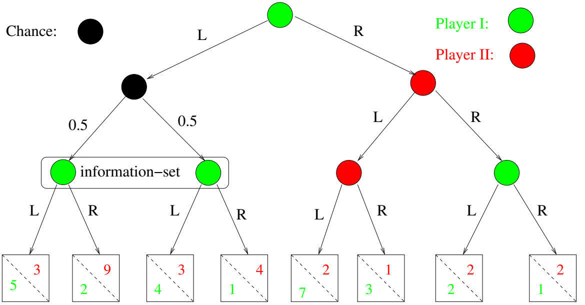
So far, we have only seen games in "strategic form" (also called "normal form"), where all players choose their strategy simultaneously (independently).
What if, as is often the case, the game is played by a sequence of moves over time? (Think, e.g., Chess.)
Consider the following 2-person game tree:



- How do we analyze and compute “solutions” to such *extensive form* games?
- What is their relationship to strategic form games?

chance, and information

Some tree nodes may be *chance* (probabilistic) nodes, controlled by neither player. (Poker, Backgammon.)
Also, a player may not be able to distinguish between several of its "positions" or "nodes", because not all *information* is available to it. (Think Poker, with opponent's cards hidden.) Whatever move a player employs at a node must be employed at all nodes in the same "information set".

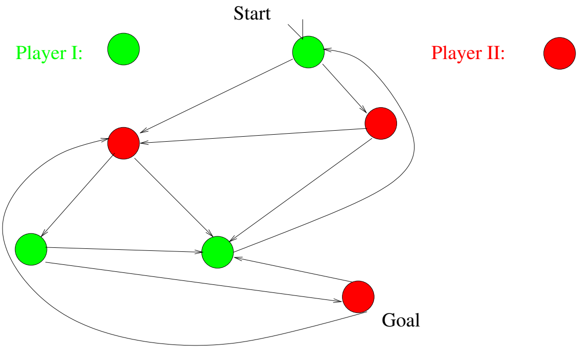


A game where every information set has only 1 node is called a game of *perfect information*.

Theorem Any finite n-person extensive game of perfect information has an “equilibrium in pure strategies”.

Again, how do we compute equilibrium solutions for such games?

Extensive games on Graphs



- Does Player I have a strategy to "force" the play to reach the "Goal"?
- Such games have lots of applications.
- Again, how do we compute winning strategies in such games?
- What if some nodes are chance nodes?

Mechanism Design

Suppose you are the game designer. How would you design the game so that the "solutions" will satisfy some "objectives"?

- **Example:** Auctions: (think EBay, or Google Ads) Think of an auction as a multiplayer game between several bidders. If you are the auctioneer, how could you design the auction rules so that, for every bidder, bidding the maximum that an item is worth to them will be a "dominant strategy"?
  A answer: second price, sealed bid *Vickrey auctions*.
- How would you design protocols (such as network protocols), to encourage "cooperation" (e.g., diminish congestion)?
- Many computational questions arise in the study of "good" mechanisms for various goals.
- This is an extremely active area of research (we will only get to scratch its surface).

But why study this stuff?

GT is a core foundation of mathematical economics.
But what does it have to do with Computer Science? More than you might think: GT ideas have played an important role in CS:

- Games in AI: modeling "rational agents" and their interactions. (Similar to Econ. view.)
- Games in Modeling and analysis of reactive systems: computer-aided verification: formulations of model checking via games, program inputs viewed "adversarially", etc.
- Games in Algorithms: several GT problems have a very interesting algorithmic status (e.g., in NP, but not known to be NP-complete, etc).
- Games in Logic in CS: GT characterizations of logics, including modal and temporal logics (Ehrenfeucht-Fraisse games and bisimulation).

- Games in Computational Complexity: Many computational complexity classes are definable in terms of games: Alternation, Arthur-Merlin games, the Polynomial Hierarchy, etc.. Boolean circuits, a core model of computation, can be viewed as games (between AND and OR).

More recently:

- Games, the Internet, and E-commerce: An extremely active research area at the intersection of CS and Economics. Basic idea: "The internet is a HUGE experiment in interaction between agents (both human and automated)". How do we set up the rules of this game to harness "socially optimal" results?

I hope you are convinced: knowledge of the principles and algorithms of game theory will be useful to you for carrying on future work in many CS diciplines.

Ok, let's get started

Definition A *strategic form game* Γ, with n players, consists of:

1. A set $N = \{1, \ldots, n\}$ of players.
2. For each $i \in N$, a set S_i of *(pure) strategies*.
   Let $S = S_1 \times S_2 \times \ldots \times S_n$ be the set of possible combinations of (pure) strategies.
3. For each $i \in N$, a *payoff (utility) function* $u_i : S \mapsto \mathbb{R}$, describes the payoff $u_i(s_1, \ldots, s_n)$ to player i under each combination of strategies.

(Each player prefers to maximize its own payoff.)

Definition A zero-sum game Γ, is one in which for all $s = (s_1, \ldots, s_n) \in S$,

$$u_1(s) + u_2(s) + \ldots + u_n(s) = 0.$$

Some "food for thought"

Food for Thought (**the "guess half the average game"**): Consider a strategic-form game Γ with *n*-players. Each player has to guess a whole number from 1 to 1000. The player who guesses a number that is closest to half of the average guess of all players wins a payoff of 1. All other players get a payoff of 0. (If there are ties for who is closest, those who are closest share the payoff of 1 equally amongst themselves; alternatively, all who are closest get payoff 1.)

Question: What would your strategy be in such a game?

Question: What is a "Nash Equilibrium" of such a game?