Algorithmic Game Theory and Applications

Lecture 16: Selfish Network Routing, Congestion Games, and the Price of Anarchy

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games and the internet

- Basic idea: "The internet is a huge experiment in interaction between agents (both human and automated)".
- Such interactions can profitably be viewed from a game theoretic viewpoint: agents trying to maximize their own payoffs, etc.
- What are the implications of selfish behavior?
- How do we set up the rules of these games to harness "socially optimal" results?

(Selfish) Network Routing as a Game

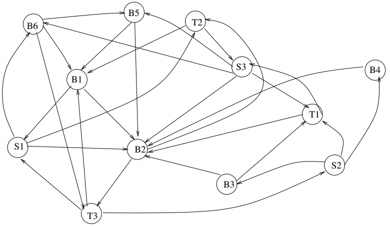


Figure: "The Internet"

▷ Selfish agent $i = 1, 2, 3$, wants to route packets from Si to Ti. So agent i must choose a **directed path** from Si to Ti.
▷ The **delay** on each edge of the path is governed by the **congestion** of that edge, i.e., the total number of agents using that edge in their path.
▷ Agents can change their choice to decrease their delay.
▷ What is a **Nash Equilibrium** in this game? What are the **welfare properties** of such an NE? (Is it socially optimal? If not, how bad can it be?)

Congestion Games ([Rosenthal,1973])

A **Congestion Game**, $G = (N, R, (Z_i)_{i \in N}, (d_r)_{r \in R})$ has:
$\triangleright$ A finite set $N = \{1, \ldots, n\}$ of **players**.
$\triangleright$ A finite set of $R = \{1, \ldots, m\}$ of **resources**.
$\triangleright$ For each player, i, a set $Z_i \subseteq 2^R$, of admissible **strategies** for player i. So a pure strategy, $s_i \in Z_i$ is a set of resources.
$\triangleright$ Each resource $r \in R$ has a **cost function**: $d_r : \mathbb{N} \rightarrow \mathbb{Z}$. Intuitively, $d_r(j)$ is the cost of using resource r if there are j agents simultaneously using r.
$\triangleright$ For a pure strategy profile $s = (s_1, \ldots, s_n) \in Z_1 \times \ldots Z_n$, the **congestion** on resource r is: $n_r(s) \doteq |\{i \mid r \in s_i\}|$.
$\triangleright$ Under strategy profile $s = (s_1, \ldots, s_n)$, the **total cost** to player i is:

$$C_i(s) \doteq \sum_{r \in s_i} d_r(n_r(s))$$

$\triangleright$ Every player, i, wants to <u>*minimize*</u> its own (expected) cost.

Best response dynamics, and pure Nash Equilibria

In a congestion game G, for any pure strategy profile $s = (s_1, \ldots, s_n)$, suppose that some player i has a better alternative strategy, $s'_i \in Z_i$, such that $C_i(s_{-i}; s'_i) < C_i(s)$.

Player i can switch (unilaterally) from s_i to s'_i. This takes us from profile s to profile (s_{-i}, s'_i).

We call this a single (strict) improvement step.

Starting at an arbitrary pure strategy profile s, what happens if the players perform a sequence of such improvement steps?

Theorem: ([Rosenthal'73]) *In any congestion game, every sequence of strict improvement steps is necessarily finite, and terminates in a pure Nash Equilibrium.*
Thus, in particular, every congestion game has a pure strategy Nash Equilibrium.

Proof: Potential functions

Proof: Consider the following **potential function**:

$$\varphi(s) \doteq \sum_{r \in R} \sum_{i=1}^{n_r(s)} d_r(i) \qquad (1)$$

What happens to the value of $\varphi(s)$ if player i switches unilaterally from s_i to s_i', taking profile s to $s' := (s_{-i}; s_i')$?

Claim: $\varphi(s) - \varphi(s') = C_i(s) - C_i(s')$.
Proof: Re-order the players in any arbitrary way, and index them as players $1, 2, \ldots, n$. (In particular, any player formerly indexed i could be re-indexed as n.) For $i' \in \{1, \ldots, n\}$, define

$$n_r^{(i')}(s) = |\{i \mid r \in s_i \wedge i \in \{1, \ldots, i'\}\}|$$

By exchanging the order of summation in equation (1) for $\varphi(s)$, it can be seen that (check this yourself):

$$\varphi(s) = \sum_{i=1}^{n} \sum_{r \in s_i} d_r(n_r^{(i)}(s))$$

proof of claim, continued

Now note that $n_r^{(n)}(s) = n_r(s)$. Thus

$$\sum_{r \in s_n} d_r(n_r^{(n)}(s)) = \sum_{r \in s_n} d_r(n_r(s)) = C_n(s)$$

So, if player n switches from strategy s_n to s'_n, leading us from profile s to $s' = (s_{-n}; s'_n)$, then:
$\varphi(s) - \varphi(s') = C_n(s) - C_n(s')$.
But note that when re-ordering we could have chosen player n to be any player we want! So this holds for every player i. $\square$

Proof of Theorem, continued

To complete the proof of Rosenthal's Theorem: Observe that every strict improvement step must decreases the value of the potential function $\varphi(s)$ by at least 1 (the costs $d_r(s)$ are all integers). Furthermore, there are only finitely many pure strategies s, so there are finite integers: $a = \min_s \varphi(s)$ and $b = \max_s \varphi(s)$. Thus, every improvement sequence is finite.

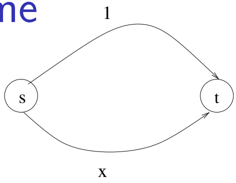
Finally, note that the last profile s in any improvement sequence which can not be further improved is, by definition, a pure Nash equilibrium.$\square$

Complexity of pure NE in network conges. games

Consider a **network congestion game** where we are given a network with source-sink node pairs (S_i, T_i), for each player i, and each player must to choose a route (path) from S_i to T_i. Suppose the cost (**delay**) of an edge, e, under profile s, is defined to be some *linear* function: $d_e(n_e(s)) = \alpha_e n_e(s) + \beta_e$.

One obvious way to compute a pure NE is to perform an arbitrary improvement sequence. However, this may conceivably require many improvement steps.
Is there a better algorithm?
It turns out that it is as hard as any **polynomial local search** problem to compute a pure NE for network congestion games:
Theorem: [Fabrikant et.al.'04, Ackermann et.al.'06].
Computing a pure NE for a network congestion game is **PLS-complete**, even when all edge delay functions, d_e, are *linear*.
So, unfortunately, a P-time algorithm is unlikely.

A flow network game



(from [Roughgarden-Tardos'00])

▷ n customers in network: each wants to go from s to t.

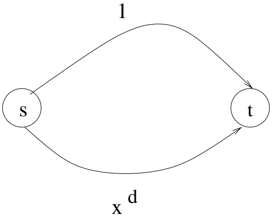
▷ Each can either take the edge with "latency" 1 (delay of crossing the edge), or edge with latency x. Here x represents the "congestion": the ratio of the number of customers that are using that edge divided by the total n.

▷ Assume n is **very large**, (basically, $n \rightarrow \infty$). What is the delay in Nash Equilibrium? (NEs in such a setting yield essentially a unique average delay [Beckmann, et. al. '56].)

▷ What is a "globally optimal" customer routing strategy profile that minimizes average delay?

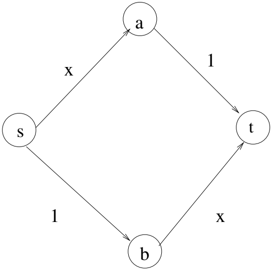
What is the globally optimal average delay?

a modified game



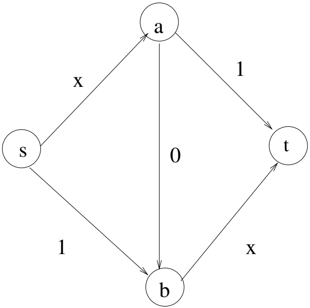
- What is the NE, and what is the average delay it induces?
- What is the globally optimal average delay?

a different network



- What is the NE, and what is its average delay?
- What is a globally optimal strategy profile and optimal average delay?
- What if an ambitious “network service provider” wanted to build an additional “high capacity superfast broadband” line?

Braess's paradox



- What is the NE and its average delay?
- What is the globally optimal average delay?

social welfare and the price of anarchy

Recall that in a strategic game Γ, "utilitarian social welfare", $welfare(x)$, under a particular profile of mixed strategies $x \in X$, is defined as $welfare(x) := \sum_{i=1}^{n} U_i(x)$. For a game Γ, let $NE(\Gamma)$ be the set of NE's of Γ.

For our next definition suppose $welfare(x) > 0$ for all $x \in X$. (In many games, we can enforce this by, e.g., adding a fixed value to all payoffs.)

A version of "the price of anarchy" can be defined as: ([Koutsoupias-Papadimitriou'98])

$$PA(\Gamma) := \frac{\max_{x \in X} welfare(x)}{\min_{x \in \mathrm{NE}(\Gamma)} welfare(x)}$$

Thus, the "price of anarchy" is the ratio of best "global" outcome to the the worst NE outcome. Note: this ratio is ≥ 1 and larger means "worse".

It would be comforting to establish that in various situations the "price of anarchy" isn't too high.

Pure price of anarchy

In some settings, such as congestion games, where we know that a pure equilibrium exists, it is sometimes more sensible to compare the best overall outcome to the worst pure-NE outcome.
Let pure-$NE(\Gamma)$ denote the set of pure NEs in the game Γ. For settings (such as congestion games) where we know pure-$NE(\Gamma)$ is non-empty, we define "the pure price of anarchy" as:

$$\text{pure-}PA(\Gamma) := \frac{\max_{s \in S} welfare(s)}{\min_{s \in \text{pure-NE}(\Gamma)} welfare(s)}$$

Thus, the "pure price of anarchy" is the ratio of best (pure) "global" outcome to the the worst pure NE outcome.

price of anarchy in the flow network game

- For flow f let $welfare(f) := 1/(\text{average s-t-delay})$.
- In Braess's paradox, the price of anarchy is $4/3$: by playing the NE the average delay is 2, but playing half-and-half on the upper and lower route, the average delay is $3/2$ (and that's optimal).
- We have seen that the price of anarchy in network games can be arbitrarily high, when x^d is an edge label.
- Remarkably, [Roughgarden-Tardos'00] showed that in a more general flow network setting (where there can be multiple source-destination pairs (s_j, t_j)), as long as "congestions" labeling edges are linear functions of x, the worst-case price of anarchy is $4/3$.
- In other words, for linear latencies, the Braess's paradox example yields the worst-case scenario.

Back to atomic network congestion games

By an "atomic" network congestion game, we simply mean a standard network congestion game with a *finite* number of players, where each aims to minimize its own cost. (Whereas in non-atomic network flow games the average cost in equilibrium is uniquely determined, this is not the case with atomic network congestion games.)
What is the (pure) price of anarchy in atomic network congestion games?

Theorem: [Christodoulou-Koutsoupias'2005]. *The pure price of anarchy for a pure NE in atomic network congestion games with linear utilities is*

$$5/2$$

(And this is tight, just like 4/3 for non-atomic network congestion games.)