Algorithmic Game Theory and Applications

Lecture 14: Simulation, Bisimulation, and other Ehrenfeucht-Fräisse Games

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Simulation

- A game played on two labelled directed graphs $G_1 = (V_1, E_1, l_1)$ and $G_2 = (V_2, E_2, l_2)$, whose nodes are labelled by a symbol from alphabet Σ. Namely, $l_i : V_i \mapsto \Sigma$, for $i = 1, 2$. We assume that each vertex of both graphs has outdegree at least 1 (this just simplifies the game description).

- Initially, the game starts with a pebble on a start node u_0 of G_1 and a start node v_0 of G_2.

- In iteration i, player 1 picks a vertex u_i, such that $(u_{i-1}, u_i) \in E_1$, and player 2 responds by picking a vertex v_i such that there is an edge $(v_{i-1}, v_i) \in E_2$.

- Player 1 wins the game if it is ever the case that $l_1(u_i) \neq l_2(v_i)$, for any iteration i. Player 2 wins the game otherwise.

- This is a win-lose game of perfect information. By Borel determinacy (in fact, by open set determinacy), the game is determined, and one player or the other has a winning strategy.

Bisimulation

- Same as simulation, except for one thing:
- in each round, i, player 1 gets to choose whether to pick the next vertex v_i or the next vertex u_i, and player 2 has to respond by picking u_i or v_i, respectively.
- Obviously, this win-lose game is also determined for similar reasons.
- These games are important in logic/automata theory.

  In particular, bisimulation captures the expressive power of *propositional modal logic* in the following sense: two vertices u_0 and v_0 of two labelled directed graphs G_1 and G_2 are not distinguishable by any propositional modal formula if and only if player 2 has a winning strategy in the bisimulation game over G_1 and G_2 starting at u_0 and v_0.
- Given G_1 and G_2, and u_0 and v_0, how can we efficiently decide which player has a winning strategy in this game? (Hint: you already know the answer from previous lectures.)

Ehrenfeucht-Fräisse Games and First-Order Logic

- Just as (bi)simulation captures the expressive power of modal logic, there are games that capture the expressive power of other logics.

- In particular, first-order logic, which can arguably be called "the mother of all logics", is captured by Ehrenfeucht-Fraisse Games.

  Recall: a first-order formula looks like, e.g.:

  $\forall x\, \exists y\, \forall z (E(x,y) \wedge \neg E(y,z)) \vee (E(y,x) \wedge E(x,y))$

- We will stick to Ehrenfeucht-Fräisse games played on a pair of directed graphs $G_0 = (V_1, E_1)$ and $G_1 = (V_2, E_2)$. The game definition generalizes naturally to games played on arbitrary first-order structures.

- in the k-pebble EF-game, each player has k pebbles. These pebbles come in named pairs $(P_{0,i}, P_{1,i})$, $i = 1, \ldots, k$, respectively.

  In each round, Player 1 chooses some $i \in \{1, \ldots, k\}$, and picks up one of the two pebbles

$P_{j,i}$, where j is either 0 or 1, and it places $P_{j,i}$ on some vertex v of G_j. Then Player 2 responds by picking up $P_{1-j,i}$ and placing it on some vertex v' of G_{1-j}.

- Player 1 wins if it is ever the case that the mapping which maps the vertex pebbled by $P_{0,i}$ to the vertex pebbled by $P_{1,i}$, for $i = 1, \ldots, k$, is not an isomorphism of the "induced subgraph" induced in G_0 by the vertices pebbled by $P_{0,1}, \ldots, P_{0,k}$, and that induced in G_1 by $P_{1,1}, \ldots, P_{1,k}$.

- **Theorem:** (Ehrenfeucht'61) *The two structures G_0 and G_1 are indistinguishable by a first-order formula with k variables if and only if player 2 has a winning strategy in the k-pebble EF-game on G_0 and G_1.*

- Given G_1 and G_2, how would can we decide if there is any first-order formula with k variables that distinguishes them?