Algorithmic Game Theory and Applications

Lecture 12: Games on Graphs

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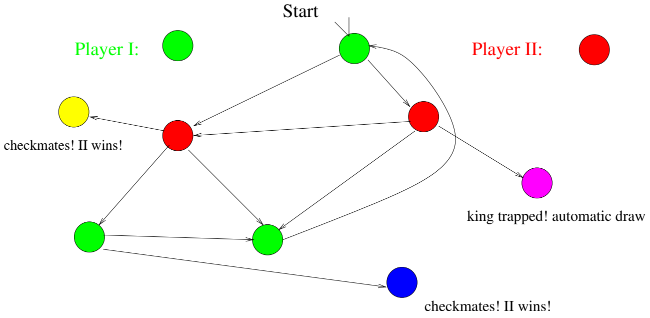
unbounded chess



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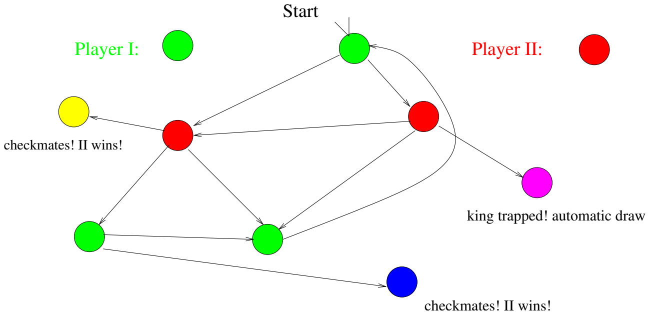
unbounded chess



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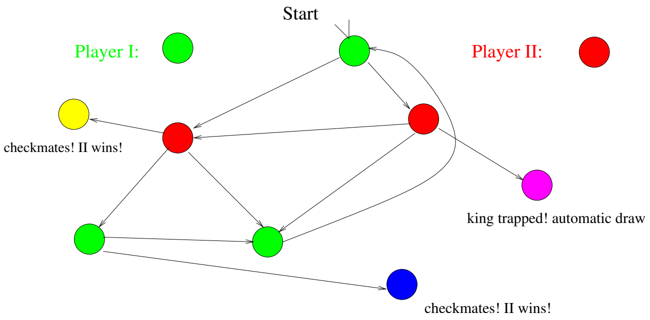
▷ There are finitely many positions ($\leq 64^{32}$). After some depth, every "play" contains recurrences of positions.

unbounded chess



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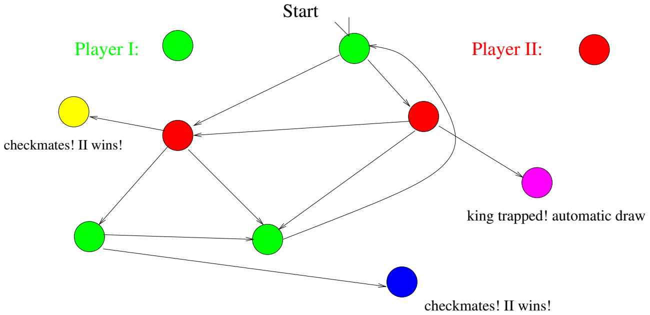
▷ Consider "<u>unbounded chess</u>" without artificial stopping conditions: an infinite play is by definition a <u>draw</u>.

unbounded chess



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▷ There are finitely many positions ($\leq 64^{32}$). After some depth, every "play" contains recurrences of positions.
▷ Consider "unbounded chess" without artificial stopping conditions: an infinite play is by definition a draw.

Is this win-lose-draw game determined? I.e., does Zermelo's theorem still hold?

more serious motivation

- ▷ We can often model the dynamics of a system (e.g., a running program) as a state transition system.
- ▷ If the system interacts with an environment, transitions out of some states can be viewed as "controlled by the environment". Can the environment force the system, with some sequence of inputs, into a "bad state"?
- ▷ Even for state machines without environments, certain temporal queries about the behavior of the system over time can be formulated as a game on a graph.
- ▷ Such queries, and much more, can be formalized in certain "temporal logics": formal languages for describing relationships between the occurrence of events over time. Efficiently checking such queries against a system model (e.g., a state transition system) is the task of "model checking". Some key model checking tasks are intimately related to efficiently solving certain games on graphs.

game graphs and their trees

A 2-player **game graph**, $G = (V, E, pl)$ consists of:

$\triangleright$ A (finite) set V of vertices.

$\triangleright$ A set $E \subseteq V \times V$ of edges.

$\triangleright$ A partition (V_1, V_2) of the vertices $V = V_1 \cup V_2$ into two disjoint sets belonging to players 1 and 2, respectively.

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A game graph G together with a start vertex $v_0 \in V$, defines a game tree T_{v_0} given by:

$\triangleright$ Action alphabet $\Sigma = V$. Thus $T_{v_0} \subseteq V^*$.

$\triangleright$ $\epsilon \in T_{v_0}$, and $wv'' \in T_{v_0}$, for $v'' \in V$, if and only if

- $w = \epsilon$ and $(v_0, v'') \in E$, or
- $w = w'v'$, for some $v' \in V$, and $(v', v'') \in E$.

We extend the partition (V_1, V_2) to a partition (T_1', T_2') of the tree nodes of T_{v_0} as follows: if $v_0 \in V_i$, then $\epsilon \in T_i'$, and if $v' \in V_i$, then any tree node $wv' \in T_i$.

T_{v_0} is thus a game tree, where $Act(wv') = \{v'' \mid (v', v'') \in E\}$, whose plays are all paths in the graph G starting from v_0.

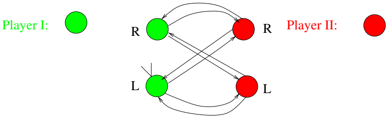
games on graphs

A **game on a graph**, $\mathcal{G}_{v_0}$, is given by:

> *A finite game graph G, vertex $v_0 \in V$, and payoff function $u : \Psi_{T_{v_0}} \mapsto \mathbb{R}$.*

These together define a 2-player zero-sum PI-game with game tree T_{v_0}.

Note: We already know that even for win-lose payoff functions u, games on finite graphs are not in general determined, because the infinite binary tree $\{L, R\}^*$ is the game tree for the following game graph:



and we already know (lecture 11) that there are sets Y of plays such that the win-lose game $\langle \{L, R\}^*, Y \rangle$ is not determined.

So, let's restrict the possible payoff functions.

"history oblivious" payoffs

▷ Suppose ∃ vertex v' of graph G that is a "dead end". E.g., in chess this could be "checkmate for Player I".
▷ There may be many ways to get to v', but the winner is the same for any finite play $wv' \in V^*$. I.e., $u(wv') = u(w'v')$, for all $wv', w'v' \in V^*$. So, the payoff is "history oblivious".
▷ What about for infinite plays π? We can think of π as an infinite sequence $v_0 v_1 v_2 v_3 v_4 v_5 \ldots\ldots$, where each $v_i \in V$.
We use the notation $\pi \in V^\omega$.
▷ For $\pi = v_0 v_1 \ldots$, let

$$\inf(\pi) = \{v \in V \mid \text{for } \infty\text{-many } i \in \mathbb{N},\ v_i = v\}$$

▷ Let's call payoff function $u()$ history oblivious (**h.o.**), if for all infinite plays π & π', if $\inf(\pi) = \inf(\pi')$, then $u(\pi) = u(\pi')$, and for all finite complete plays wv and $w'v$,

$$u(wv) = u(w'v).$$

Call a graph game h.o. if its payoffs are h.o.
We will only consider h.o. games (and often less).

“finitistic” payoffs

▷ Note that in chess, if the play π is infinite, then the play is always a draw, i.e., $u(\pi) = 0$.
▷ Let's call an h.o. payoff function finitistic if for all infinite plays π and π', $u(\pi) = u(\pi')$. Let's call a game on a graph $\mathcal{G}_{v_0}$ finitistic if its payoff function is.
So, in win-lose-draw finitistic games, infinite plays are either all wins, all losses, or all draws, for player 1.
Question: Are all finitistic games on graphs determined?
Answer: Yes.............
In fact, more it true: for finitistic games there is always a memoryless strategy for each player that achieves the value of the game, and we can efficiently compute these strategies.

memoryless strategies and determinacy

Definition For a game $\mathcal{G}_{v_0}$, a strategy s_i for player i is a <u>**memoryless strategy**</u> if for all $wv, w'v \in Pl'_i$, $s_i(wv) = s_i(w'v)$, and if $wv_0 \in Pl'_i$ then $s_i(wv_0) = s_i(\epsilon)$.
I.e., the strategy always makes the same move from a vertex, regardless of the history of how it got there.
Let MLS_i denote the set of memoryless strategies for player i.
MLS_i is a finite set, even if S_i is not. In particular, if $m = |Pl_i|$ is the number of vertices belonging to player i, then $|MLS_i| \leq |\Sigma|^m$.

Definition $\mathcal{G}_{v_0}$ is **memorylessly determined** if both players have memoryless strategies that achieve "the value". I.e.,

$$\max_{s_1 \in MLS_1} \inf_{s_2 \in S_2} u(s_1, s_2) = \min_{s_2 \in MLS_2} \sup_{s_1 \in S_1} u(s_1, s_2)$$

Theorem A Finitistic games on finite graphs are memorylessly determined. Moreover, there is an efficient (P-time) algorithm to compute memoryless value-achieving strategies in such games.

the win-lose case: easy "fixed point" algorithm

We first prove the theorem for finitistic win-lose games via an easy "bottom up" fixed point algorithm.

Input: Game graph $G = (V, E, pl, v_0)$.

Assume w.l.o.g. all infinite plays are win for player 2 (other case is symmetric). "Dead end": vertex with no outgoing edge. $Good := \{v \in V \mid v \text{ a dead end that wins for player } 1\}$. $Bad := \{v \in V \mid v \text{ a dead end that wins for player } 2\}$.

1. Initialize: $Win_1 := Good$; $St_1 := \emptyset$;
2. **Repeat**

   Foreach $v \notin Win_1$:

   If $(pl(v) = 1 \;\; \& \;\; \exists\, (v, v') \in E : \;\; v' \in Win_1)$

   $Win_1 := Win_1 \cup \{v\}$; $St_1 := St_1 \cup \{v \mapsto v'\}$;

   If $(pl(v) = 2 \;\; \& \;\; \forall\, (v, v') \in E : \;\; v' \in Win_1)$

   $Win_1 := Win_1 \cup \{v\}$;

   Until The set Win_1 does not change;

Player 1 has a Win.-Strategy iff $v_0 \in Win_1$. If so, St_1 is a memoryless winning strategy for player 1.

why does this work?

Proof of Theorem A: (for the win-lose case)
$\triangleright$ First, we claim that for each $v \in Win_1$, St_1 is a winning strategy for player 1 in the game $\mathcal{G}_v$ (i.e., the game that starts at node v).
Suppose $v \in Win_1$. It must have entered Win_1 after, say, m iterations of the repeat loop. By induction on m, if player 1 plays according to (partial) strategy St_1, then it is guaranteed a win in the game $\mathcal{G}_v$ within m moves. Note that St_1 may be partial: it may only tell us how to move from some vertices. This won't matter.
Base case: $m = 0$, $v \in Good$.
Inductively: either v is player 1's vertex or 2's.
If it is player 1's, then $St_1(v) = v'$, where $(v, v') \in E$ and $v' \in Win_1$, and furthermore v' entered Win_1 by $m - 1$ iterations. By induction St_1 wins for player 1 from v' in $m - 1$ moves.

If v is player 2's, then we know that <u>for all</u> $(v, v') \in E$, $v' \in Win_1$, and furthermore v' entered Win_1 by $\leq m - 1$ iterations. Thus, no matter what move player 2 makes, in 1 move, by induction, we will be at a vertex $v' \in Win_1$ where player 1 wins with St_1 within $m - 1$ moves.

$\triangleright$ Now consider $v \notin Win_1$ when algorithm halts. For each $v' \in pl^{-1}(2)$, if $\exists\,(v', v'') \in E$, with $v'' \notin Win_1$, then pick one such v'', and let $St_2 := St_2 \cup \{v' \mapsto v''\}$. St_2 may also be partial.

We claim St_2 is a memoryless winning strategy for player 2 in every game $\mathcal{G}_v$, where $v \notin Win_1$. Suppose St_2 is not a winning strategy for some $v \notin Win_1$. Then player 1 must be able to win by reaching a *Good* vertex within say, m moves from v against St_2. Let's show this is a contradiction.

Base case: m$=0$, but then $v \in Good$. $\Rightarrow\Leftarrow$.

Inductively: either v is player 1's or player 2's.

If player 1's, then $\forall (v, v') \in E$, $v' \notin Win_1$, because otherwise by the algorithm $v \in Win_1$. Suppose player 1's winning strategy is to play $(v, v') \in E$. It must have a win within $m-1$ moves from $v' \notin Win_1$ against St_2. $\Rightarrow\Leftarrow$.

If it is player 2's move, then one possibility is $v \in \mathit{Bad}$, $(\Rightarrow\Leftarrow)$.
Otherwise, $St_2(v) = v'$ must be defined: since $v \notin \mathit{Win}_1$, there must exist $(v, v') \in E$ with $v' \notin \mathit{Win}_1$. Otherwise, by the algorithm, $v \in \mathit{Win}_1$.

By induction, player 1 must have a $(m-1)$-winning strategy from $v' \notin \mathit{Win}_1$. $\Rightarrow\Leftarrow$.

generalizing to finitistic zero-sum

The generalization is not hard:
In a finitistic game, there can only be a bounded number, $r \leq |V| + 1$, of distinct payoffs $u(\pi)$,

$$j_1 < j_2 < j_3 < \ldots < j_r$$

and one of these, say j_k, is the payoff $u(\pi)$ for all infinite plays π. Suppose, w.l.o.g., that $k < r$. (If instead $1 < k$, then we work symmetrically with respect to player 2. If $1 = k = r$, then all payoffs are equal and there is nothing to do.)
Consider a new win-lose game where player 1 wins if it attains payoff j_r, and loses if its payoff is any less. Use the fixed point algorithm on this game to find a memoryless (partial) strategy for player 1 that is winning from vertices in Win_1 where payoff j_r can be obtained. We can then eliminate Win_1 vertices and the payoff j_r. We get a new finitistic zero-sum game, with payoffs $j_1 < \ldots < j_{r-1}$. Repeat!!

non-finitistic win-lose h.o. games: Muller games

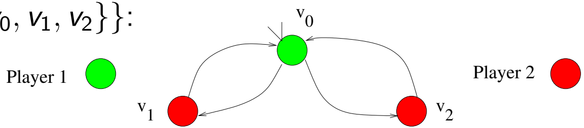
▷ We will only be interested in win-lose h.o. games.
By attaching a "self-loop" to every dead-end vertex, every play becomes infinite, and we can define the "payoffs" via a set $\mathcal{F} \subseteq 2^V$, where

$$\mathcal{F} = \{F \subseteq V \mid \text{player 1 wins if } \inf(\pi) = F\}$$

We call $\mathcal{F}$ the (Muller) winning condition. Let's call such win-lose h.o. games Muller games.

▷ **Question:** Are all Muller games determined? **Answer:** Yes.

▷ **Question:** Are all Muller games memorylessly determined?
Answer: No! Consider the following Muller game,
$\mathcal{F} = \{\{v_0, v_1, v_2\}\}$:



Does Player 1 have a winning strategy?
Does it have a memoryless winning strategy?

remarks

▷ Muller games and restricted variants of them are important in applications to model checking. We can't do them full justice here.

▷ Every Muller game can be converted to an "equivalent" (but potentially exponentially larger) game with a limited kind of Muller winning condition called a parity condition. These so called parity games *are* memorylessly determined.

▷ Can we find winning strategies in parity games efficiently (in P-time)? This is a tantalizing open problem.

It follows from memoryless determinacy that finding winning strategies for them is in **NP** $\cap$ **co-NP**: we can guess a memoryless strategy for either player and efficiently verify that it is a winning strategy.

▷ An older survey text on all this is:
[*"Automata, Logics, and Infinite Games",*
edited by E. Grädel, W. Thomas, T. Wilke, 2002].

food for thought: back to LP

Consider the following LP, for solving a finitistic win-lose game with game graph G. (Suppose w.l.o.g., player 1 loses if the play is infinite.) Let $V = \{v_1, \ldots, v_n\}$ be vertices of G. We will have one LP variable x_i for each vertex $v_i \in V$.

Minimize x_m
Subject to:
$0 \leq x_i \leq 1$, *for* $i = 1, \ldots, n$;
$x_i = 1$, *for* v_i *a winning dead end for player 1.*
$x_i = 0$, *for* v_i *a losing dead end for player 1.*
For each x_i *where* $\mathrm{pl}(v_i) = 1$,
$\quad x_i \geq x_j$, *for each* $(v_i, v_j) \in E$.
For each x_i *where* $\mathrm{pl}(v_i) = 2$,
$\quad$ *and* $\{v_{j_1}, \ldots, v_{j_r}\} = \{v' \mid (v_i, v') \in E\}$,
$\quad x_i \leq x_{j_k}$, *for* $k = 1, \ldots, r$, *and*
$\quad x_i \geq x_{j_1} + \ldots + x_{j_r} - (r - 1)$

$\triangleright$ The optimal value of the given LP is 1 iff player 1 has a winning strategy in $\mathcal{G}_{v_m}$.

$\triangleright$ Now, what if instead of 2 players, player 1 was playing "alone against nature/chance"? Could you formulate an LP for 1's optimal payoff? This would be a simple instance of a "Markov Decision Process".