Algorithmic Game Theory and Applications

Lecture 10: Games in Extensive Form

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the setting and motivation

▷ Most games in “real life” are not in “strategic form”: players don't pick their entire strategies independently (“simultaneously”). Instead, the game transpires over time, with players making “moves” to which other players react with “moves”, etc. <u>Examples:</u> chess, poker, bargaining, dating, ...

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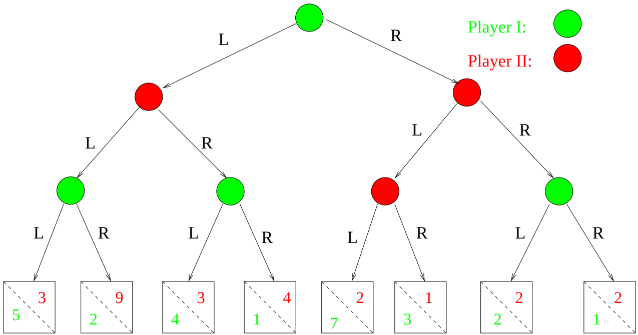
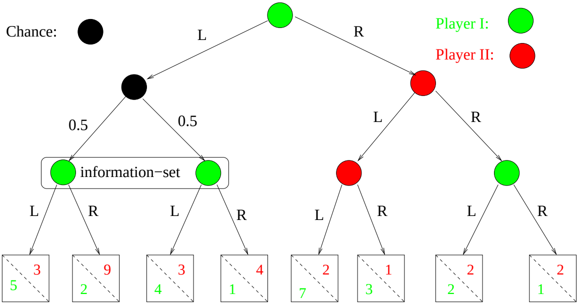
▷ A "game tree" looks something like this:



▷ But we may also need some other "features".

chance, information, etc.

Some tree nodes may be *chance* (probabilistic) nodes, controlled by no player (by "nature"). (Poker, Backgammon.) Also, a player may not be able to distinguish between several of its "positions" or "nodes", because not all *information* is available to it. (Think Poker, with opponent's cards hidden.) Whatever move a player employs at a node must be employed at all nodes in the same "information set".



To define extensive form games, we have to formalize all these.

Trees: a formal definition

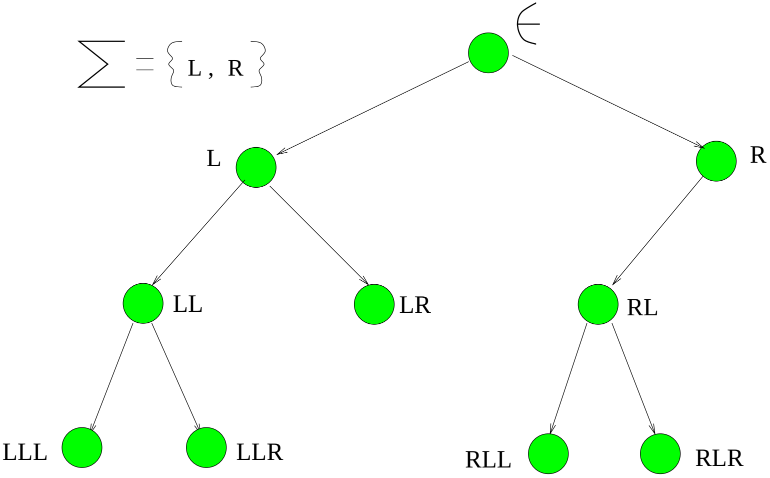
▷ Let $\Sigma = \{a_1, a_2, \ldots, a_k\}$ be an alphabet. A tree over Σ is a set $T \subseteq \Sigma^*$, of nodes $w \in \Sigma^*$ such that: if $w = w'a \in T$, then $w' \in T$. (I.e., it is a prefix-closed subset of Σ^*.)

▷ For a node $w \in T$, the children of w are $ch(w) = \{w' \in T \mid w' = wa \text{ , for some } a \in \Sigma\}$. For $w \in T$, let $Act(w) = \{a \in \Sigma \mid wa \in T\}$ be "actions" available at w.

▷ A leaf (or terminal) node $w \in T$ is one where $ch(w) = \emptyset$. Let $L_T = \{w \in T \mid w \text{ a leaf}\}$.

▷ A (finite or infinite) path π in T is a sequence $\pi = w_0, w_1, w_2, \ldots$ of nodes $w_i \in T$, where if $w_{i+1} \in T$ then $w_{i+1} = w_i a$, for some $a \in \Sigma$. It is a complete path (or play) if $w_0 = \epsilon$ and every non-leaf node in π has a child in π. Let Ψ_T denote the set of plays of T.

Tree example



games in extensive form

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3. A partition of the tree nodes T into disjoint sets $Pl_0, Pl_1, \ldots, Pl_n$, such that $T = \bigcup_{i=0}^{n} Pl_i$. Where Pl_i, $i \geq 1$, denotes the nodes of player i, where it is player i's turn to move. (And Pl_0 denotes the "chance"/"nature" nodes.)

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6. For each player i, a function $u_i : \Psi_T \mapsto \mathbb{R}$, from (complete) plays to the payoff for player i.

explanation and comments

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▷ We defined our alphabet of possible actions Σ to be finite, which is generally sufficient for our purposes. In other words, the tree is finitely branching. In more general settings, even the set of possible actions at a given node can be infinite.

▷ Later, we will focus on the following class of games:
Definition An extensive form game $\mathcal{G}$ is called a game of **perfect information**, if every information set $Info_{i,j}$ contains only 1 node.

pure strategies

$\triangleright$ A <u>pure strategy</u> s_i for player i in an extensive game $\mathcal{G}$ is a function $s_i : Pl_i \mapsto \Sigma$ that assigns actions to each of player i's nodes, such that $s_i(w) \in Act(w)$, & such that if $w, w' \in Info_{i,j}$, then $s_i(w) = s_i(w')$.
Let S_i be the set of pure strategies for player i.
$\triangleright$ Given pure profile $s = (s_1, \ldots, s_n) \in S_1 \times \ldots \times S_n$, <u>if there are no chance nodes</u> (i.e., $Pl_0 = \emptyset$) then s uniquely determines a play π_s of the game: players move according their strategies:

- Initialize $j := 0$, and $w_0 := \epsilon$;
- While (w_j is not at a terminal node)
  If $w_j \in Pl_i$, then $w_{j+1} := w_j\ s_i(w_j)$;
  $j := j + 1$;
- $\pi_s = w_0, w_1, \ldots$

$\triangleright$ What if there are chance nodes?

pure strategies and chance

If there are chance nodes, then $s \in S$ determines a probability distribution over plays π of the game.

For finite extensive games, where T is finite, we can calculate the probability $p_s(\pi)$ of play π, using probabilities $q_w(a)$:

Suppose $\pi = w_0, \ldots, w_m$, is a play of T. Suppose further that for each $j < m$, if $w_j \in Pl_i$, then $w_{j+1} = w_j\ s_i(w_j)$. Otherwise, let $p_s(\pi) = 0$.

Let $w_{j_1}, \ldots, w_{j_r}$ be the chance nodes in π, and suppose, for each $k = 1, \ldots, r$, $w_{j_k+1} = w_{j_k} a_{j_k}$, i.e., the required action to get from node w_{j_k} to node w_{j_k+1} is a_{j_k}. Then

$$p_s(\pi) := \prod_{k=1}^{r} q_{w_{j_k}}(a_{j_k})$$

For infinite extensive games, defining these distributions in general requires much more elaborate definitions (proper "measure theoretic" probability). We will avoid the heavy stuff.

chance and expected payoffs

For a finite extensive game, given pure profile $s = (s_1, \ldots, s_n) \in S_1 \times \ldots \times S_n$, we can, define the "expected payoff" for player i under s as:

$$h_i(s) := \sum_{\pi \in \Psi_t} p_s(\pi) * u_i(\pi)$$

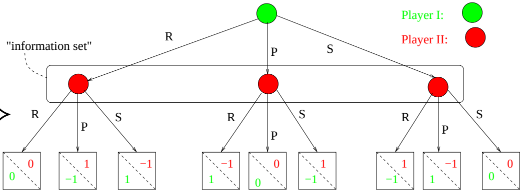
Again, for infinite games, much more elaborate definitions of "expected payoffs" would be required.
Note: This "expected payoff" does not arise because any player is mixing its strategies. It arises because the game itself contains randomness.
We can also combine both: players may also randomize amongst their strategies, and we could then define the overall expected payoff.

from strategic to extensive games

Every finite strategic game Γ can be encoded easily and concisely as an extensive game $\mathcal{G}_\Gamma$. We illustrate this via the Rock-Paper-Scissor 2-player game (the n-player case is an easy generalization):

		Player II		
		Rock	Paper	Scissors
Player I	Rock	0, 0	−1, 1	1, −1
	Paper	1, −1	0, 0	−1, 1
	Scissors	−1, 1	1, −1	0, 0

$\Longrightarrow$



from extensive to strategic games

Every extensive game $\mathcal{G}$ can be viewed as a strategic game $\Gamma_{\mathcal{G}}$:
▷ In $\Gamma_{\mathcal{G}}$, the strategies for player i are the mappings $s_i \in S_i$.
▷ In $\Gamma_{\mathcal{G}}$, we define payoff $u_i(s) := h_i(s)$, for all pure profiles s.
If the extensive game $\mathcal{G}$ is finite, i.e., tree T is finite, then the strategic game $\Gamma_{\mathcal{G}}$ is also finite.
Thus, all the theory we developed for finite strategic games also applies to finite extensive games.
Unfortunately, the strategic game $\Gamma_{\mathcal{G}}$ is generally exponentially bigger than $\mathcal{G}$. Note that the number of pure strategies for a player i with $|Pl_i| = m$ nodes in the tree, is in the worst case $|\Sigma|^m$.
So it is often unwise to naively translate a game from extensive to strategic form in order to "solve" it.
If we can find a way to avoid this blow-up, we should.

imperfect information & "perfect recall"

▷ An extensive form game (EFG) is a game of **imperfect information** if it has non-trivial (size > 1) information sets. Players don't have full knowledge of the current "state" (current node of the game tree).

▷ Informally, an imperfect information EFG has **perfect recall** if each player i never "forgets" its own sequence of prior actions and information sets. I.e., a EFG has perfect recall if whenever $w, w' \in \mathit{Info}_{i,j}$ belong to the same information set, then the "*visible history*" for player i (sequence of information sets and actions of player i during the play) prior to hitting node w and w' must be exactly the same.

▷ [Kuhn'53]: with perfect recall it suffices to restrict players' strategies to **behavior strategies**: strategies that only randomize (independently) on actions at each information set.

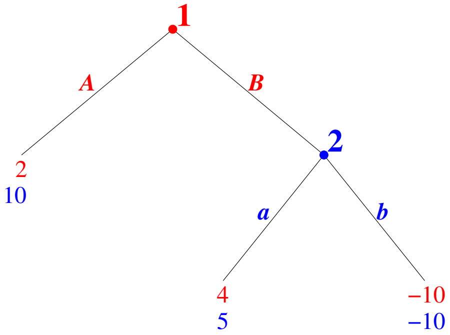
▷ Perfect recall is often assumed as a "sanity condition" for EFGs (most games we encounter do have perfect recall).

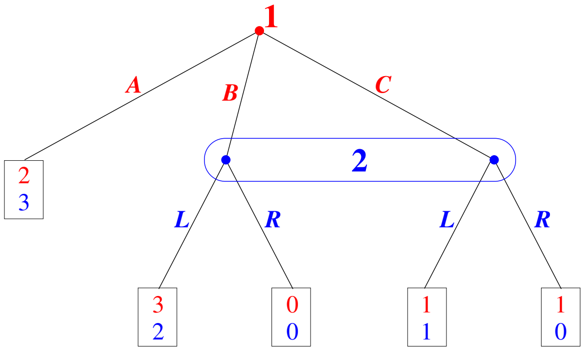
subgames and (subgame) perfection

▷ A **subgame** of an extensive form game is any subtree of the game tree which has *self-contained information sets*. (I.e., every node in that subtree must be contained in an information set that is itself entirely contained in that subtree.)

▷ For an extensive form game G, a profile of behavior strategies $b = (b_1, \ldots, b_n)$ for the players is called a **subgame perfect equilibrium** (SGPE) if it defines a Nash equilibrium for *every* subgame of G.

▷ [Selten'75]: Nash equilibrium (NE) (and even SPGE) is inadequately refined as a solution concept for extensive form games. In particular, such equilibria can involve **"Non-credible threats"**:


1
A
B
2
10
2
a
b
4
5
−10
−10



Addressing this general inadequecy of NE and SGPE requires a more refined notion of equilibrium called **trembling-hand perfect equilibrium** [Selten'73].

solving games of imperfect info.

For EFGs *with perfect recall* there are ways to avoid the exponential blow-up of converting to normal form. We only briefly mention algorithms for imp-inf games.
(See, e.g., [Koller-Megiddo-von Stengel'94].)

▷ In strategic form 2-player zero-sum games we can find minimax solutions efficiently (P-time) via LP. For 2-player zero-sum extensive imp-info games (without perfect recall), finding a minimax solution is **NP-hard**. NE's of 2-player EFGs can be found in exponential time.

▷ The situation is better with perfect recall: 2-player zero-sum imp-info games of perfect recall can be solved in P-time, via LP, and 2-player NE's for arbitrary perfect recall games can be found in exponential time using a Lemke-type algorithm.

▷ [Etessami'2014]: For EFGs with ≥ 3 players with perfect recall, computing refinements of Nash equilibrium (including "trembling-hand perfect" and "quasi-perfect") can be reduced to computing a NE for a 3-player normal form game.

Our main focus will be games of perfect information. There the situation is much easier.

games of perfect information

A game of perfect information has only 1 node per information set. So, for these we can forget about information sets.

Examples: Chess, Backgammon, ...

counter-Examples: Poker, Bridge, ...

Theorem([Kuhn'53]) *Every finite extensive game of perfect information, $\mathcal{G}$, has a NE (in fact a SGPE) in pure strategies.*

In other words, there is a pure profile $(s_1, \ldots, s_n) \in S$ that is a Nash Equilibrium (and a subgame perfect equilibrium).

Our proof provides an efficient algorithm to compute such a pure profile, given $\mathcal{G}$, using "backward induction".

A special case of this theorem says the following:

Proposition([Zermelo'1912]) *In Chess, either:*

1. *White has a "winning strategy", or*
2. *Black has a "winning strategy", or*
3. *Both players have strategies to force a draw.*

Next time, perfect information games.