Algorithmic Game Theory and Applications: Coursework 2

PLEASE ANSWER *** ONLY 2 *** OUT OF THE FOUR QUESTIONS on this coursework. DO NOT SUBMIT ANSWERS TO MORE THAN TWO OF THE QUESTIONS (ONLY TWO WILL BE MARKED).

This homework is due at 12pm (noon), on Thursday, March 30th. Please submit your solutions online as PDF files, using the LEARN page for AGTA. (This will be via GradesScope, with similar instructions to how you submitted the PDF files for Coursework 1.) *Do not collaborate with other students on the coursework. Your solutions must be your own.*

Each question counts for 50 points, for a total of 100 points for ANSWERING *** TWO *** QUESTIONS ONLY.

1. Recall that a Nash equilibrium in an extensive form game is *subgame perfect nash equilibrium* (SPNE) if it is also a Nash equilibrium in every *subgame* of the original game. Formally, a *subgame*, is a game defined by a subtree, T_v of the original game tree, T, such that the subtree T_v, rooted at a node v, has the property that for every decendent u of v in the game tree (including v itself), every node in the same information set as u is also in the subtree T_v.

   (a) [14 points] Give an example of a <u>pure</u> NE which is *not* a SPNE, for a finite extensive form game of perfect information.

   (b) [20 points] Prove that every finite extensive game of perfect information where *there are no chance nodes* and where no player gets the same payoff at any two distinct leaves, must have a <u>*unique*</u> pure-strategy SPNE.

   (c) [16 points] Give an example of a finite extensive form game that contains a pure Nash Equilibrium but does not contain any subgame-perfect pure Nash Equilibrium. Justify your answer.

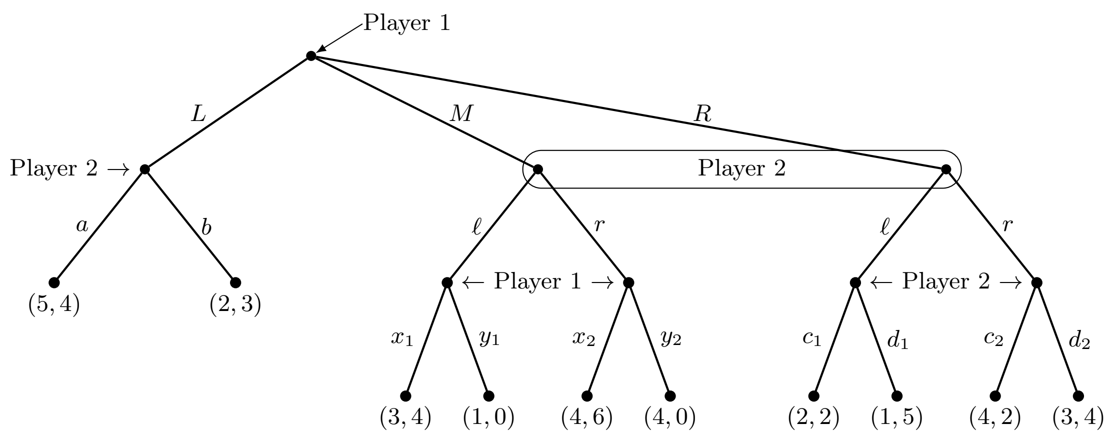

Figure 1: Question 2

2. Consider the finite extensive form game of imperfect information depicted in Figure 1. (When a leaf node is labeled by a pair (i, j) of integers, that means the payoff at that leaf to player 1 is i and the payoff to player 2 is j.)

   (a) [6 points] Does this game satisfy "perfect recall"? Explain.

   (b) [24 points] Identify *all* SPNEs in this game, in terms of "behavior strategies". Explain why what you have identified are all SPNEs.

   Next, consider the following 2-player zero-sum finite extensive form game of perfect information. There are two players, player 1 (you), and player 2 (your opponent). There are three 6-sided dice, D_1, D_2, and D_3. However, these dice are not "ordinary" dice, in the sense that their sides are not labeled with all the numbers 1 through 6. Instead:

   - Die D_1 has five of its sides labeled by the number 4 and one of its sides labeled by the number 1.
   - Die D_2 has three of its sides labeled by the number 2 and three of its sides labeled by the number 5.
   - Die D_3 has five of its sides labeled by the number 3, and one of its sides labeled by the number 6.

   All three dice D_1, D_2, and D_3, have the following usual property of an ordinary die: each time you role any of these dice, each of the six sides of that die are equally likely (with probability 1/6) to show up on top when the die has stopped rolling.

The game between player 1 and 2 is played as follows:

- Player 1 (you) has the first "move". Player 1 can either choose one of the three dice, D_1 or D_2 or D_3, or else it can "pass" and allow player 2 (the opponent) to choose one of the three dice first.
- Next, it is player 2's turn. If player 1 has already chosen a die, then player 2 must choose one of the other two remaining dice not chosen already by player 1. If instead player 1 has "passed" in the prior move, then Player 2 must now choose one of the three dice (any one). Player 2 cannot "pass".

  Afterwards, if player 1 had not already chosen a die (meaning it had passed earlier and allowed player 2 to choose first), then player 1 now has to choose one of the other two remaining dice, not chosen already by player 2.
- Afterwards, when both players have each chosen their respective die, each player roles their own die, and whoever rolls a higher number wins a "payoff" of 1 Dollar from the other player. (So, the payoff of the player who rolled a higher number is $+1$ and the payoff of the player who rolled a lower number is -1. Note that it isn't possible for both players to roll the same number, because no two dice among D_1, D_2, and D_3 have a common number labeling any one of their sides.)

Note that since this defines a finite 2-player zero-sum extensive form game of perfect information, by Kuhn's theorem there exists a pure minimax profile in this game (i.e., a pure Nash equilibrium).

(c) [20 points] Compute the minimax value of this two player zero-sum extensive form game (from the perspective of player 1 (you), the maximizer), and compute a pure minimax profile in this game (i.e., a pair of pure minmaximizer and pure maxminimizer strategies, for player 1 and player 2, respectively).

Do not draw the extensive form game tree explicitly (it is too big), and do not try to describe the pure minimax profile in terms of moves from explicit nodes on the explicit game tree (again, that would be too big and unmanagable). Instead, describe the pure strategies of the two players in the minimax profile

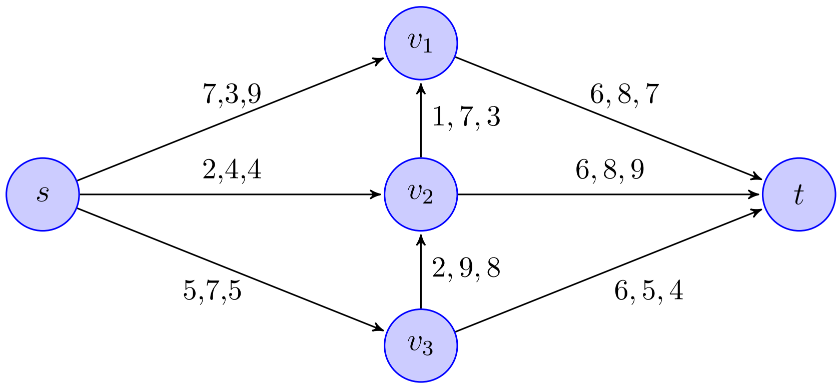

Figure 2: Question 3

more intuitively and succinctly, by describing exactly how each player should move and react to the other player's prior possible moves/choice(s). Explain why the pure strategies you have described constitute a minimax profile.

3. (a) [20 points] Recall *Rosenthal's Theorem*, namely that every finite *congestion game* has a *pure* Nash Equilibrium. In the proof we gave in the lecture slides for Rosenthal's theorem, we defined the potential function $\varphi(s)$, which for any pure strategy profile s is defined as:

$$\varphi(s) := \sum_{r \in R} \sum_{i=1}^{n_r(s)} d_r(i)$$

Later in the proof we claimed that $\varphi(s)$ can also be expressed as a different nested sum, but we didn't prove that fact, and instead said "check this yourself". This question asks you to prove that fact: Prove that for any pure strategy profile s the following equality holds:

$$\varphi(s) = \sum_{i=1}^{n} \sum_{r \in s_i} d_r(n_r^{(i)}(s))$$

(b) Consider the *atomic network congestion game*, with three players, described by the directed graph in Figure 2.

In this game, every player i (for $i = 1, 2, 3$) needs to choose a directed path from the source s to the target t. Thus, every player i's set of possible actions (i.e., its set of pure strategies) is the set of all possible directed paths from s to t.

Each edge e is labeled with a sequence of three numbers (c_1, c_2, c_3). Given a profile $\pi = (\pi_1, \pi_2, \pi_3)$ of pure strategies (i.e., s-t-paths) for all three players, the *cost* to player i of each directed edge, e, that is contained in player i's path π_i, is c_k, where k is the total number of players that have chosen edge e in their path. The total cost to player i, in the given profile π, is the sum of the costs of *all* the edges in its path π_i from s to t. Each player of course wants to minimize its own total cost.

i. [20 points] Compute a pure strategy NE in this atomic network congestion game, giving also the total cost for each player in that pure NE. Explain why what you have computed is a pure NE.

ii. [10 points] Is the pure NE you have computed in part (i.) pareto optimal in terms of costs, amongst the set of all pure strategy combinations for the players? Explain.

4. The auction house Christie's of London is auctioning a triptych (a series of three related painting) by the famous artist Fracis Bacon, entitled "Three Studies of Isabel Rawsthorne". We will refer to the three paintings in the triptych series as T1, T2, and T3, respectively.

   Suppose that Christie's hires you as an auction designer, and suppose that you decide to use the Vickery-Clarke-Groves mechanism as an auction, in order to determine which bidder should get which part(s) of the triptych, and at what price. Suppose that there are only three bidders. The three bidders' names are: Susanne (S), Lakshmi (L), and Bill (B).

   Since you are running a VCG-based auction, you ask each bidder to give you their valuation for every subset of the paintings in the triptych, as part of the bidding process. Suppose that the valuation functions v_S , v_L, and v_B that you receive from the three bidders, S, L, and B, respectively, are as follows (the numbers denote 10^5 pounds):

	valuation							
bidder i	$v_i(\emptyset)$	$v_i(T_1)$	$v_i(T_2)$	$v_i(T_3)$	$v_i(T_1, T_2)$	$v_i(T_1, T_3)$	$v_i(T_2, T_3)$	$v_i(T_1, T_2, T_3)$
$i := S$	0	16	16	13	29	36	29	54
$i := L$	0	4	7	12	38	37	37	53
$i := B$	0	10	18	4	26	28	39	54

(a) [28 points] Give a VCG outcome for this auction. In other words, specify, in the VCG outcome, which bidders will get which of the painting(s), and what price they will each pay for the painting(s) they get. Justify your answer, and show your calculations.

(b) [16 points] Is the VCG outcome you have calculated in part (a) unique? Are the VCG prices paid by the player's uniquely determined? Justify your answer, and show your calculations.

(c) [6 points] Comment on the wisdom of choosing the VCG mechanism for this or any auction. Do you think it is a good idea to do so? What if instead of this triptych, Christie's wanted to do a simultaneous auction of 20 Andy Warhol paintings, and they knew that at least 30 viable bidders want to bid for (subsets of) those paintings. Would you suggest using the VCG mechanism for such an auction? What alternative auction would you use, and why? Explain, briefly.