Algorithmic Game Theory and Applications: Homework 1

PLEASE ANSWER *** ONLY 2 *** OUT OF THE FOUR QUESTIONS on this coursework. DO NOT SUBMIT ANSWERS TO MORE THAN TWO OF THE QUESTIONS (ONLY TWO WILL BE MARKED).

This homework is due by 3:00pm, on Tuesday, February 28th.
Please submit your solutions online as PDF files, using the LEARN page for AGTA, via Gradescope. (Instructions for how to submit the PDF files on LEARN will be provided separately to all students via the LEARN page.) *Do not collaborate with other students on the coursework. Your solutions must be your own.*

Each question counts for 50 points, for a total of 100 points for ANSWERING *** TWO *** QUESTIONS ONLY. (DO NOT SUBMIT ANSWERS FOR MORE THAN TWO QUESTIONS. ONLY TWO WILL BE MARKED.)

1. (a) (25 points) Consider 2-player strategic form game, G, specified by the following "bimatrix":

$$\begin{bmatrix} (7,4) & (9,5) & (2,3) & (2,6) \\ (4,1) & (9,2) & (2,1) & (3,6) \\ (8,7) & (6,7) & (5,8) & (2,7) \\ (3,3) & (7,4) & (4,6) & (2,4) \end{bmatrix}$$

   As usual, player 1 is the row player, and player 2 is the column player. If the content of the bimatrix at row i and column j is the pair (a,b), then $u_1(i,j) = a$ and $u_2(i,j) = b$.

   Compute *all* of the Nash equilibria (NEs) of this game G, together with the expected payoff to each player in each NE. Explain why any profile x that you claim is an NE of G, is indeed an NE of G, and furthermore, explain why there are no other (pure or mixed) NEs of G, other than the profile(s) you claim are NE(s) of G.

   (b) (25 points) Consider a finite game, G, with pure strategy sets $S_1, \ldots, S_n$ for the n players, and with a payoff function $u_i(s)$ for each player $i \in \{1, \ldots, n\}$ that assigns a payoff to each pure strategy profile $s = (s_1, \ldots, s_n) \in S = S_1 \times S_2 \times \ldots \times S_n$.

   Now consider a different n-player game, G', which has exactly the same strategy sets $S_1, \ldots, S_n$, as G, but where the payoff function

$u_i'(s)$ for each player i differs from $u_i(s)$ as follows:

$$u_i'(s) = u_i(s) + g_i(s_{-i})$$

where, for each player i, $g_i : S_{-i} \rightarrow \mathbb{R}$ is a function (any function) that depends *only* on the other players' pure strategies and not on player i's own pure strategy.

Prove that G and G' have exactly the same set of Nash equilibria. In other words, prove that a mixed strategy profile $x = (x_1, \ldots, x_n)$ is a NE for G if and only if x is a NE for G'.

(Note, however, that the same NE may yield entirely different payoffs to the different players in G and in G'.)

2. (a) (20 points) Consider the 2-player zero-sum game given by the following payoff matrix for player 1 (the row player):

$$\begin{bmatrix} 5 & 6 & 4 & 7 & 3 \\ 6 & 4 & 5 & 3 & 8 \\ 7 & 2 & 5 & 4 & 6 \\ 4 & 7 & 3 & 5 & 7 \\ 3 & 5 & 8 & 6 & 3 \end{bmatrix}$$

Compute both the minimax value for this game, as well as a minimax profile (NE), i.e. a pair of minmaximizer and maxminimizer strategies for players 1 and 2, respectively.

(You can, for example, use the linear programming solver package `linprog` in MATLAB, available on DICE machines, to do this. To run MATLAB, type "matlab" at the shell command prompt. For guidance on using the `linprog` package, see: `http://uk.mathworks.com/help/optim/ug/linprog.html`.)

(b) (30 points) Recall from Lecture 7 on LP duality, the *symmetric* 2-player zero-sum game, G, for which the (skew-symmetric) payoff matrix (in block form) for player 1 is:

$$B = \begin{bmatrix} 0 & A & -b \\ -A^T & 0 & c \\ b^T & -c^T & 0 \end{bmatrix}$$

Suppose that there exist vectors $x' \in \mathbb{R}^n$ and $y' \in \mathbb{R}^m$, such that $Ax' < b$, $x' \geq 0$, $A^T y' > c$ and $y' \geq 0$. (Note the two *strict* inequalities.) Prove that for the game G, every minmaximizer strategy $w = (y^*, x^*, z)$ for player 1 (and hence also every maxminimizer strategy for player 2, since the game is symmetric) has the property that $z > 0$, i.e., the last pure strategy is played with positive probability. (Recall that this was one of the missing steps in our sketch proof in the lecture that the minimax theorem implies the LP duality theorem.)

(Hint: Let $w = (y^*, x^*, z)$ be a maxminimizer strategy for player 2 in the game G. Note that the value of any symmetric 2-player zero-sum game must be equal to zero. This implies, by the minimax theorem, that $Bw \leq 0$. Suppose, for contradiction, that $z = 0$. What does this imply about Ax^* , $A^T y^*$, and $b^T y^* - c^T x^*$? Then if $y^* \neq 0$, show that this implies $(y^*)^T (Ax' - b) < 0$. In turn, show that it also implies $(x^*)^T (A^T y' - c) > 0$. Use these and related facts to conclude a contradiction.)

3. Consider the following simple *2-player zero-sum* games, where the payoff table for Player 1 (the row player) is given by:

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

We can view this as a game where each player chooses "heads" (H) or "tails" (T), where the first strategy for each player is denoted H and the second strategy is denoted T.

   (a) (10 points) First, what is the unique Nash equilibrium, or equivalently the unique minimax profile of mixed strategies for the two players, in this game? And what is the minimax value of that game?

   (b) (40 points) Now, suppose that the two players play the same game you have chosen in part (a), against each other, over and over again, for ever, and suppose that both of them use the following method in order to update their own strategy after each round of the game.

      i. At the beginning, in the first round, each player chooses either of the pure strategies, H or T, arbitrarily, and plays that.

      ii. After each round, each player i accumulates statistics on how its opponent has played until now, meaning how many Heads and how many Tails have been played by the opponent, over all rounds of the game played thusfar. Suppose these numbers are N Heads and M Tails.
      Then player i uses these statistics to "guess" its opponents *"statistical mixed strategy"* as follows. It assumes that its opponent will next play a mixed strategy σ, which plays Heads with probability $N/(N+M)$ and plays Tails with probability $M/(N+M)$.
      Under the assumption that its opponent is playing the *"statistical mixed strategy"* σ, in the next round player i plays a pure strategy (H or T) that is a pure *best response* to σ.
      If both H and T are a best response at any round, then player i can use **any tie breaking rule it wish** in order to determine the pure strategy it plays in the next round.

      iii. They repeat playing like this forever.

Prove that, regardless how the two players start playing the game in the first round, the "statistical mixed strategies" of both players in this method of repeatedly playing the game will, in the long run, as the number of rounds goes to infinity, converge to their mixed strategies in the unique Nash equilibrium of the game.

You are allowed to show that this holds using any specific tie breaking rule that you want. Please specify the precise tie breaking rule you have used. (It turns out that it holds true for any tie breaking rule. But some tie breaking rules make the proof a lot easier than others.)

4. (a) (40 points) One variant of the Farkas Lemma says the following:

   Farkas Lemma A linear system of inequalities $Ax \leq b$ has a solution x if and only if there is no vector y satisfying $y \geq 0$ and $y^T A = 0$ (i.e., 0 in every coordinate) and such that $y^T b < 0$.

   Prove this Farkas Lemma with the aid of Fourier-Motzkin elimination. (*Hint:* One direction of the "if and only if" is easy. For the other direction, use induction on the number of columns of A, using the fact that Fourier-Motzkin elimination "works". Note basically that each round of Fourier-Motzkin elimination can "eliminate one variable" by pre-multiplying a given system of linear inequalities by a *non-negative* matrix.)

   (b) (10 points) Recall that in the *Strong Duality Theorem* one possible case (case 4, in the theorem as stated on our lecture slides) is that *both* the primal LP and its dual LP are *infeasible*. Give an example of a primal LP and its dual LP, for which both are infeasible (and argue why they are both infeasible).