Algorithms and Data Structures 2023/24
Week 7 Solutions

1. Given a flow network $\mathcal{N} = (\mathcal{G} = (V, E), c, s, t)$, let f_1 and f_2 be two flows in $\mathcal{N}$ (ie, satisfying the three flow properties wrt $\mathcal{N}$). The *flow sum* $f_1 + f_2$ is the function from $V \times V$ to $\mathbb{R}$ defined by:

$$(f_1 + f_2)(u, v) \; = \; f_1(u, v) + f_2(u, v)$$

for all $u, v \in V$.

Which of the three flow properties (wrt $\mathcal{N}$) will $f_1 + f_2$ satisfy, and which will it violate?

Answer: The three properties are *capacity constraints*, *skew-symmetry*, and *flow conservation.*

Capacity constraints: $f_1 + f_2$ might *violate* the capacity constraints. As an example, consider the network of question 2. Let f_1 be the flow shown in question 2. Let f_2 be the flow that ships 4 units along the path $s \to x \to y \to t$. Then if we add these flows directly as prescribed in this question, we will (for example) define

$$(f_1 + f_2)(y, t) = f_1(y, t) + f_2(y, t) = 4 + 4 = 8.$$

This certainly breaks the capacity constraint for (y, t) which is 4.

Skew-symmetry: $f_1 + f_2$ will *satisfy* skew-symmetry. We know f_1 and f_2 individually satisfy skew-symmetry, because they are flows. Therefore for any (u, v), we have

$$(f_1 + f_2)(u, v) \; = \; f_1(u, v) + f_2(u, v) \; = \; -f_1(v, u) - f_2(v, u) \; = \; -(f_1 + f_2)(v, u),$$

as required (using the defn of $f_1 + f_2$ and the skew-symmetry property for f_1, f_2).

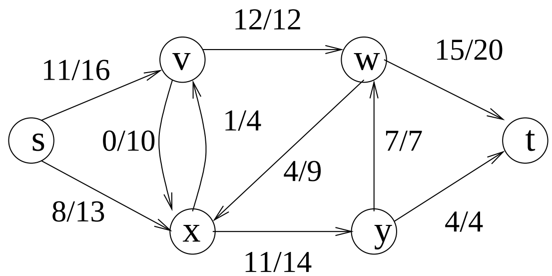
Flow conservation: $f_1 + f_2$ will *satisfy* flow conservation. Flow conservation for a flow f states that for all $u \in V \setminus \{s, t\}$, we have $\sum_{v \in V} f(u, v) = 0$. We know this holds individually for f_1, f_2. Let $u \in V \setminus \{s, t\}$. Then we can write

$$\sum_{v \in V} (f_1 + f_2)(u, v) \; = \sum_{v \in V} (f_1(u, v) + f_2(u, v)) \; = \; \sum_{v \in V} f_1(u, v) + \sum_{v \in V} f_2(u, v) = 0 + 0 = 0.$$

Hence flow conservation holds for $f_1 + f_2$.

tutors: Use this as an opportunity to point out the difference between this Q and the case when f_2 is a flow in the *residual network* (wrt f_1) - in that case everything has been set up for the capacity condition to also hold.

2. **Question:** we are given



Two questions:

(a) Find a pair of subsets $X, Y \subseteq V$ such that $f(X, Y) = -f(V - X, Y)$.

(b) Find a different pair of subsets $X, Y \subseteq V$ such that $f(X, Y) \neq -f(V - X, Y)$.

Answer: The point of this question is to get thinking about flow between *sets of vertices*, by applying Lemma 3 of Lecture slides 10-11. However, it might be good to think about specific examples of (a), (b) first, before looking at the details of what the pattern is.

What we are asking is: when is it the case that

$$f(X, Y) + f(V - X, Y) = 0?$$

Remember from Lemma 3 (part 3) of slides 10-11 that for any two *disjoint* sets $X', Y' \subset V$, and any other set Z', and any flow f, we have $f(X', Z') + f(Y', Z') = f(X' \cup Y', Z')$. Observe that for our question, certainly X and $V - X$ are disjoint sets. Hence by Lemma 3 (3), we know

$$f(X, Y) + f(V - X, Y) = f(X \cup (V - X), Y) = f(V, Y).$$

So we are testing whether $f(V, Y) = 0$ for (a), and whether $f(V, Y) \neq 0$ for (b) - once this is satisfied, X can be anything...

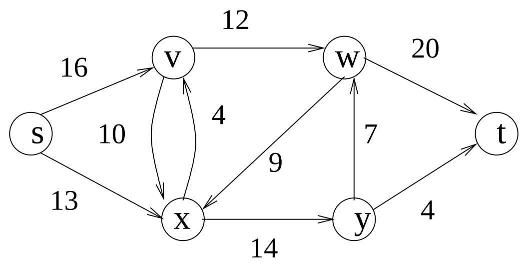
To make $f(V, Y) = 0$, we should either take Y such that $Y \cap \{s, t\} = \emptyset$, or $Y \cap \{s, t\} = \{s, t\}$. This can be seen by repeated application of part (3) of Lemma 3 from slides 10-11. To make $f(V, Y) \neq 0$, we should take Y such that $|Y \cap \{s, t\}| = 1$.

Here are some concrete examples of this behaviour:

(a) As a concrete example, let $Y = \{v, x\}$. X can be *any* set, take $X = \{w\}$ as an example. Then $f(X, Y) = -12 + 4 = -8$. Then $f(V - X, Y) = 11 + 8 - 11 = 8$.

(b) As a concrete example, take $Y = \{s\}$. Take $X = \{w\}$ again. Then we have $f(X, Y) = 0$. We have $f(V - X, Y) = -11 - 8 = -19$.

3. **Question:** execute the Ford-Fulkerson algorithm (*using the Edmonds-Karp heuristic*) on the Network below:

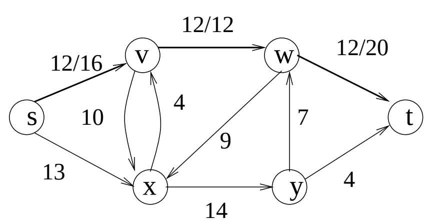


Answer: If we are using the Edmonds-Karp heuristic, then every time we search for an augmenting path, we must choose a shortest augmenting path.

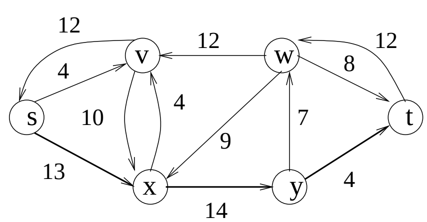
For our given network, we can see that on the first iteration, the path $p1 = s \to v \to w \to t$ is a shortest path. We have $c(p1) = 12$. Hence we define the flow $f1 = f_{p1}$ by

$$f1(e) = f_{p1}(e) = \begin{cases} 12 & \text{for } e = (s,v),(v,w),(w,t) \\ -12 & \text{for } e = (v,s),(w,v),(t,w) \\ 0 & \text{otherwise} \end{cases}$$

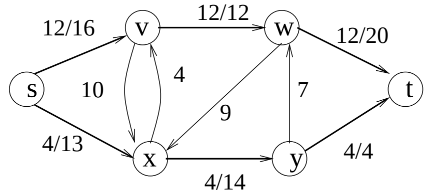
Pictorially, we have



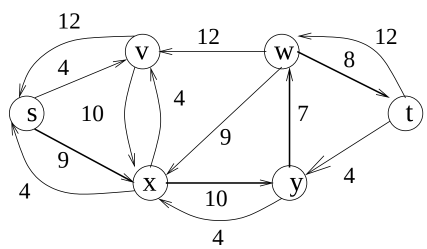
The *residual network* $\mathcal{N}_{f1}$ is as follows:



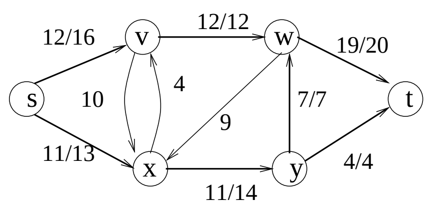
We now examine $\mathcal{N}_{f1}$ to find a shortest augmenting path. We find that $p2 = s \to x \to y \to t$ is a shortest augmenting path in $\mathcal{N}_{f1}$, min capacity 4, see above.... We therefore define a new flow f_{p2} such that 4 units are shipped along the edges of the path $p2$, and -4 shipped in the backwards direction of $p2$. Then we define the flow $f2 = f1 + f_{p2}$. Remember to point out this is possible *only* because $f1$ is a flow in $\mathcal{N}$ and $f2$ is a flow in the *residual* network $\mathcal{N}_{f1}$. Below is the flow $f2 = f1 + f_{p2}$ in $\mathcal{N}$.



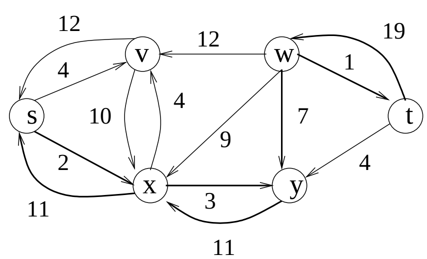
Below is the residual network $\mathcal{N}_{f2}$. If we again try the Edmonds-Karp rule for finding an augmenting path of shortest possible length, we find the path $p3 = s \to x \to y \to w \to t$ (this is of length 4, but there are no paths of length 3 or less in $\mathcal{N}_{f2}$). The min capacity along the path is 7.



We define a new flow f_{p3} in $\mathcal{N}_{f2}$ by shipping 7 units along $p3$. Then we define the flow $f3$ in $\mathcal{N}$ as $f3 = f2 + f_{p3}$. The flow looks as follows:



We compute the residual network $\mathcal{N}_{f3}$, see below for a picture.



By Ford-Fulkerson's algorithm, we now try for a (shortest) augmenting path in the $\mathcal{N}_{f3}$. However, if we examine $\mathcal{N}_{f3}$, we see that there is *no* augmenting path from s to t - the set of vertices accessible from s is now $\{s, v, x, y\}$.

Hence we terminate, returning the flow $f3$, of value 23.

4. **Question:** A well-known problem in graph theory is the problem of computing a *maximum matching* in a *bipartite graph* $\mathcal{G}$. Give an algorithm which shows how to solve this problem in terms of the network flow problem.

   Definitions:
   A (undirected) graph $\mathcal{G} = (V, E)$ is *bipartite* if we have $V = L \cup R$ for two disjoint sets L, R, such that for every edge $e = (u, v)$ exactly one of the vertices u, v lies in L, and the other in R.
   A *matching* in an (undirected) graph G is a collection M of edges, $M \subseteq E$, such that for every vertex $v \in V$, v belongs to *at most one* edge of M.
   A *maximum matching* is a matching of maximum cardinality (for a specific graph).

   Answer:

   To solve this question, we will design a network, based on the bipartite graph $\mathcal{G}$, where a maximum flow in the network corresponds to a maximum matching in $\mathcal{G}$.

   Define the vertex set V' for our network $\mathcal{N}$ to be $V' = L \cup R \cup \{s, t\}$, where s, t are two new distinguished vertices.

   Define the (directed) edge set E' as follows:

$$E' = \{(s, u) : u \in L\} \cup \{(u, v) : u \in L, v \in R, (u, v) \in E\} \cup \{(v, t) : v \in R\}.$$

   notice that the middle set in the union above is just the edge set E of the original graph, with all of these edges now directed from L to R.

   Define the capacities of the network as follows:

$$\begin{aligned} c(s, u) &= 1 && \text{for every } u \in L \\ c(u, v) &= 1 && \text{for every } u \in L, v \in R, (u, v) \in E \\ c(v, t) &= 1 && \text{for every } v \in R \end{aligned}$$

   I now claim that every flow of value k in $\mathcal{N}$ corresponds to a matching of cardinality k in G. The max flow = maximum matching follows directly from this.

   $\Rightarrow$ Suppose f is a flow of value k in $\mathcal{N}$. We assume without any loss of generality that f is an integral flow (because all capacities are integers).

   Recall that in $\mathcal{N}$, the vertex s has $|L|$ neighboring edges (s, u). By definition of the value of a flow, $k = \sum_{u \in V} f(s, u) = \sum_{u \in L} f(s, u)$. Therefore exactly k of the (s, u) edges carry 1 unit of flow each (since no (s, u) edge can carry more than 1).
   Moreover by Lemma 11 in Lecture slides 13-14, every (S, T) cut in the network must be carrying flow of value k. Hence if we take $S = \{s\} \cup L$, then we see there are exactly k (u, v) edges in the network which carry exactly 1 unit of flow from left to right (since no (u, v) edge can carry more than this).

   Define $M = \{(u, v) \in E : f(u, v) = 1 \text{ in } \mathcal{N}\}$. Certainly $|M| = k$. I now show that M is a matching. For every $u \in L$, the flow conservation property must hold. For this

network, this means that for every $u \in L$, we require $(\sum_{v \in R} f(u, v)) + f(u, s) = 0$. Therefore if $f(s, u) = 0$, we require $f(u, v) = 0$ for every $(u, v) \in E$.
If $f(s, u) = 1$ (so $f(u, s) = -1$), we require $f(u, v) = 1$ for exactly *one* $(u, v) \in E$ (using our integer assumption). Hence every $u \in L$ will appear *at most once* in M. We can use a similar argument to show that every $v \in R$ can appear at most once in M. Hence M is a matching.

$\Leftarrow$ This is easier. Just explain how the matching of G gets mapped to $\mathcal{N}$ and check flow conservation.