Algorithms and Data Structures: Average-Case Analysis of Quicksort

Quicksort

Divide-and-Conquer algorithm for sorting an array. It works as follows:

1. If the input array has less than two elements, nothing to do ... Otherwise, do the following **partitioning** subroutine: Pick a particular key called the **pivot** and divide the array into two subarrays as follows:

$\leq$ pivot	piv.	$\geq$ pivot

2. Sort the two subarrays recursively.

Quicksort Algorithm

Algorithm $\text{QUICKSORT}(A, p, r)$

if $p < r$ **then**
    $q \leftarrow \text{PARTITION}(A, p, r)$
    $\text{QUICKSORT}(A, p, q-1)$
    $\text{QUICKSORT}(A, q+1, r)$

Partitioning

Algorithm $\text{PARTITION}(A, p, r)$

$pivot \leftarrow A[r]$
$i \leftarrow p - 1$
for $j \leftarrow p$ **to** $r - 1$ **do**
    if $A[j] \leq pivot$ **then**
        $i \leftarrow i + 1$
        exchange $A[i]$, $A[j]$
exchange $A[i+1]$, $A[r]$
return $i + 1$

Same version as [CLRS]

Analysis of Quicksort

- The **size** of an instance (A, p, r) is $n = r - p + 1$.
- Basic operations for sorting are **comparisons of keys**. We let

$$C(n)$$

be the *worst-case number of key-comparisons* performed by $\textsc{Quicksort}(A, p, r)$. We shall try to determine $C(n)$ as precisely as possible.
- It is easy to verify that the worst-case running time $T(n)$ of $\textsc{Quicksort}(A, p, r)$ is $\Theta(C(n))$ if a single comparison requires time $\Theta(1)$.
(i.e., for $\textsc{Quicksort}$, comparisons *dominate* the running time).
In any case,

$$T(n) = \Theta(C(n) \cdot \text{cost per comparison}).$$

Analysis of $\textsc{Partition}$

- $\textsc{Partition}(A, p, r)$ does *exactly* $n - 1$ comparisons for *every* input of size n.
This is of course apart from any comparisons which may be done inside the recursive calls to $\textsc{Quicksort}$.

Worst-case Analysis of $\textsc{Quicksort}$

- We get the following recurrence for $C(n)$:

$$C(n) = \begin{cases} 0 & \text{if } n \leq 1 \\ \max_{1 \leq k \leq n} \big(C(k-1) + C(n-k)\big) + (n-1) & \text{if } n \geq 2 \end{cases}$$

- Intuitively, worst-case seems to be $k = 1$ or $k = n$, i.e., everything falls on one side of the partition. This happens, e.g., if the array is sorted.

Worst-Case Analysis (cont'd)

- *Lower Bound:* $C(n) \geq \frac{1}{2}n(n+1) = \Omega(n^2)$.
Proof: Consider the situation where we are presented with an array which is already sorted. Then on every iteration, we split into one array of length $(n-1)$, and one of length 0.

$$\begin{aligned} C(n) &\geq C(n-1) + (n-1) \\ &\geq C(n-2) + (n-2) + (n-1) \\ &\;\;\vdots \\ &\geq \sum_{i=1}^{n-1} i = \frac{1}{2}n(n-1). \end{aligned}$$

- *Upper Bound:* $C(n) \leq O(n^2)$.
Bit harder (must consider all possible inputs). By induction on n, using the recurrence. Case distinction whether $k \geq n/2$.
- Overall, we will show

$$C(n) = \Theta(n^2).$$

Best-Case Analysis

- $B(n)$ = number of comparisons done by QUICKSORT in the best case.
- *Recurrence:*

$$B(n) = \begin{cases} 0 & \text{if } n \leq 1 \\ \min_{1\leq k\leq n}\big(B(k-1)+B(n-k)\big)+(n-1) & \text{if } n \geq 2 \end{cases}$$

- Intuitively, the best case is if the array is always partitioned into two parts of the same size. This would mean

$$B(n) \approx 2B(n/2) + \Theta(n),$$

which implies $B(n) = \Theta(n \lg(n))$.

Average-Case Analysis

- $A(n)$ = number of comparisons done by QUICKSORT on average if all input arrays of size n are considered equally likely.
- **Intuition:** The average case is closer to the best case than to the worst case, because only **repeatedly very unbalanced** partitions lead to the worst case.
- *Recurrence:*

$$A(n) = \begin{cases} 0 & \text{if } n \leq 1 \\ \sum_{k=1}^{n}\frac{1}{n}\big(A(k-1)+A(n-k)\big)+(n-1) & \text{if } n \geq 2 \end{cases}$$

- *Solution:*

$$A(n) \approx 2n\ln(n).$$

Average Case Analysis in Detail

We shall prove that for all $n \geq 1$ ("sufficiently large") we have

$$A(n) \leq 2\ln(n)(n+1). \qquad (\star)$$

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Average Case Analysis in Detail (cont'd)

Applying $(\star\star)$ to $(n-1)$ for $n \geq 3$, we obtain

$$(n-1)A(n-1) = 2\sum_{k=0}^{n-2} A(k) + (n-1)(n-2).$$

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Subtracting this equation from $(\star\star)$ (when $n \geq 3$)

$$nA(n) - (n-1)A(n-1) = 2A(n-1) + n(n-1) - (n-1)(n-2),$$

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and therefore

$$\frac{A(n)}{n+1} = \frac{A(n-1)}{n} + \frac{2n-2}{n(n+1)} \leq \frac{A(n-1)}{n} + \frac{2}{n}$$

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We now apply unfold-and-sum to this recurrence (stopping at $n = 2$):

$$\begin{aligned} \frac{A(n)}{n+1} &\leq \frac{A(n-1)}{n} + \frac{2}{n} \\ &\vdots \end{aligned}$$

Average Case Analysis in Detail (cont'd)

$$\frac{A(n)}{n+1} \leq \frac{A(n-2)}{n-1} + \frac{2}{n} + \frac{2}{n-1}$$

Average Case Analysis in Detail (cont'd)

$$\begin{aligned} \frac{A(n)}{n+1} &\leq \frac{A(n-2)}{n-1} + \frac{2}{n} + \frac{2}{n-1} \\ &\vdots \\ &\leq \frac{A(2)}{3} + 2\sum_{k=3}^{n} \frac{1}{k} \end{aligned}$$

Average Case Analysis in Detail (cont'd)

$$\begin{aligned} \frac{A(n)}{n+1} &\leq \frac{A(n-2)}{n-1} + \frac{2}{n} + \frac{2}{n-1} \\ &\vdots \\ &\leq \frac{A(2)}{3} + 2\sum_{k=3}^{n} \frac{1}{k} \\ &= \frac{3}{3} + 2\sum_{k=3}^{n} \frac{1}{k} \quad = \quad 2\sum_{k=2}^{n} \frac{1}{k}. \end{aligned}$$

Average Case Analysis in Detail (cont'd)

$$\begin{aligned}\frac{A(n)}{n+1} &\le \frac{A(n-2)}{n-1} + \frac{2}{n} + \frac{2}{n-1}\\ &\vdots\\ &\le \frac{A(2)}{3} + 2\sum_{k=3}^{n}\frac{1}{k}\\ &= \frac{3}{3} + 2\sum_{k=3}^{n}\frac{1}{k} \quad = \quad 2\sum_{k=2}^{n}\frac{1}{k}.\end{aligned}$$

It is easy to verify this result by induction. Thus

$$\frac{A(n)}{n+1} \le 2\sum_{k=2}^{n}\frac{1}{k} = 2\sum_{k=1}^{n-1}\frac{1}{k+1} \le 2\int_1^n \frac{1}{x} = 2\ln(n).$$

Multiplying by $(n+1)$ completes the proof of $(\star)$.

Improvements

- Use insertion sort for small arrays.
- Iterative implementation.

Main Question

Is there a way to avoid the bad worst-case performance, and in particular the bad performance on sorted (or almost sorted) arrays?

Different strategies for choosing the pivot-element help (in practice).

Median-of-Three Partitioning

Idea: Use the median of the first, middle, and last key as the pivot.

Algorithm $\textsc{M3Partition}(A, p, r)$

1. exchange $A[(p+r)/2]$, $A[r-1]$
2. **if** $A[p] > A[r-1]$ **then** exchange $A[p]$, $A[r-1]$
3. **if** $A[p] > A[r]$ **then** exchange $A[p]$, $A[r]$
4. **if** $A[r-1] > A[r]$ **then** exchange $A[r-1]$, $A[r]$
5. $\textsc{Partition}(A, p+1, r-1)$

Note that $\textsc{M3Partition}(A, p, r)$ only requires 1 more comparison than $\textsc{Partition}(A, p, r)$

Median-of-Three Partitioning (cont'd)

Algorithm $\textsc{M3Quicksort}(A, p, r)$

1. **if** $p < r$ **then**
2. $\quad q \leftarrow \textsc{M3Partition}(A, p, r)$
3. $\quad \textsc{M3Quicksort}(A, p, q-1)$
4. $\quad \textsc{M3Quicksort}(A, q+1, r)$

In can be shown that the worst-case running time of $\textsc{M3Quicksort}$ is still $\Theta(n^2)$, but at least in the case of an almost sorted array (and in most other cases that are relevant in practice) it is very efficient.

Randomized Quicksort

Idea: Use key of random element as the pivot.

Algorithm $\text{RPARTITION}(A, p, r)$

1. $k \leftarrow \text{RANDOM}(p, r)$ $\quad \triangleright$ choose k randomly from $\{p, \dots, r\}$
2. exchange $A[k]$, $A[r]$
3. $\text{PARTITION}(A, p, r)$

Algorithm $\text{RANDOMIZED QUICKSORT}(A, p, r)$

1. **if** $p < r$ **then**
2. $\quad q \leftarrow \text{RPARTITION}(A, p, r)$
3. $\quad \text{RANDOMIZED QUICKSORT}(A, p, q - 1)$
4. $\quad \text{RANDOMIZED QUICKSORT}(A, q + 1, r)$

Analysis of Randomized Quicksort

The running time of RANDOMIZED QUICKSORT on an input of size n is a **random variable**.

An analysis similar to the average case analysis of QUICKSORT shows:

Theorem

For all inputs (A, p, r), the **expected number of comparisons** *performed during a run of* RANDOMIZED QUICKSORT *on input (A, p, r), is at most $2\ln(n)(n + 1)$, where $n = r - p + 1$.*

Corollary

Thus the **expected running time** *of* RANDOMIZED QUICKSORT *on any input of size n is $\Theta(n \lg(n))$.*

Reading Assignment

Sections 7.2, 7.3, 7.4 of [CLRS] (edition 2 or 3)

Problems

1. Convince yourself that PARTITION works correctly by working a few examples, or (better) try to prove that it works correctly.
2. In our proof of the Average-running time $A(n)$, we can think of the input as being some permutation of $(1, \dots, n)$, and assume all permutations are equally likely. Why does this explain the $1/n$ factor in the recurrence on slide 10?
3. Show that if the array is initially in decreasing order, then the running time is $\Theta(n^2)$.
   $O(n^2)$ from slide 8. $\Omega(n^2)$ involves considering PARTITION on a decreasing array.