Tensor Visualisation

Visualisation – Lecture 14

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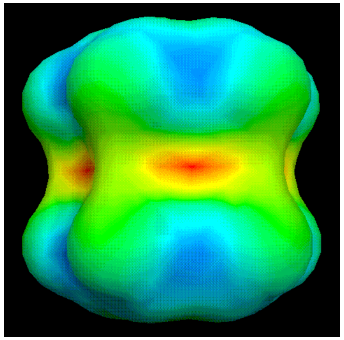
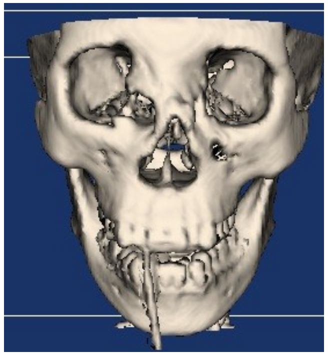
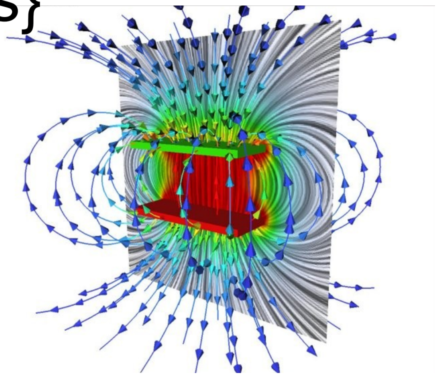
Reminder : Attribute Data Types

- **Scalar**
  - colour mapping, contouring





- **Vector**
  - lines, glyphs, stream {lines | ribbons | surfaces}



- **Tensor**
  - complex problem : active area of research
  - today : **simple techniques for tensor visualisation**

What is a tensor ?

- A tensor is a data of rank *k* defined in *n*-dimensional space ($\mathbb{R}^n$)
  - generalisation of vectors and matrices in $\mathbb{R}^n$
    - Rank 0 is a scalar
    - Rank 1 is a vector
    - Rank 2 is a matrix
    - Rank 3 is a regular 3D array
  - *k* : rank defines the **topological dimension** of the attribute
    - Topological Dimension: number of independent continuous variables specifying a position within the topology of the data
  - *n* : defines the ***geometrical dimension*** of the attribute
    - i.e. k indices each in range $0 \rightarrow (n-1)$

Tensors in $\mathbb{R}^3$

- Here we limit of discussion to tensors in $\mathbb{R}^3$
  - In $\mathbb{R}^3$ a tensor of rank k requires 3^k numbers
    - A tensor of rank 0 is a scalar ($3^0 = 1$)
    - A tensor of rank 1 is a vector ($3^1 = 3$)
    - A tensor of rank 2 is a 3x3 matrix (9 numbers)
    - A tensor of rank 3 is a 3x3x3 cube (27 numbers)

$$V = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \qquad T = \begin{bmatrix} T_{11} & T_{21} & T_{31} \\ T_{12} & T_{22} & T_{32} \\ T_{13} & T_{23} & T_{33} \end{bmatrix}$$

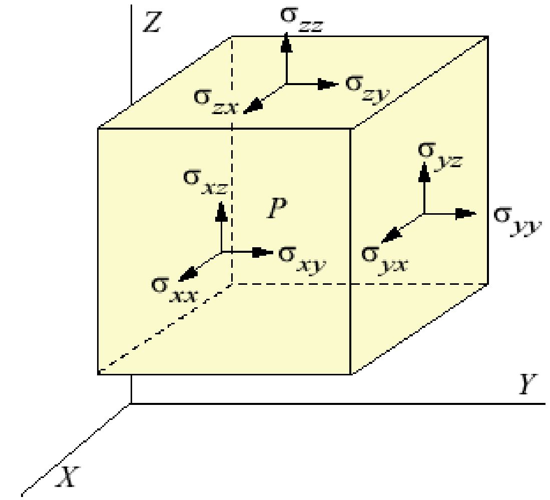
- We will only treat rank 2 tensors – i.e. matrices

Where do tensors come from?

- **Stress/strain tensors**
  - analysis in engineering
- **DT-MRI**
  - molecular diffusion measurements
- *These are represented by 3x3 matrices*
  - *Or three orthogonal eigenvectors and three corresponding eigenvalues*

Stress Tensor

- Say we are to apply force from various directions to a small box
- The stress at the surface can be represented by the **stress tensor:**
  - σ_{xx},σ_{yy} σ_{zz} indicates a **'normal' stress** in x,y,z direction, respectively
  - The rest indicates a **shear stress**



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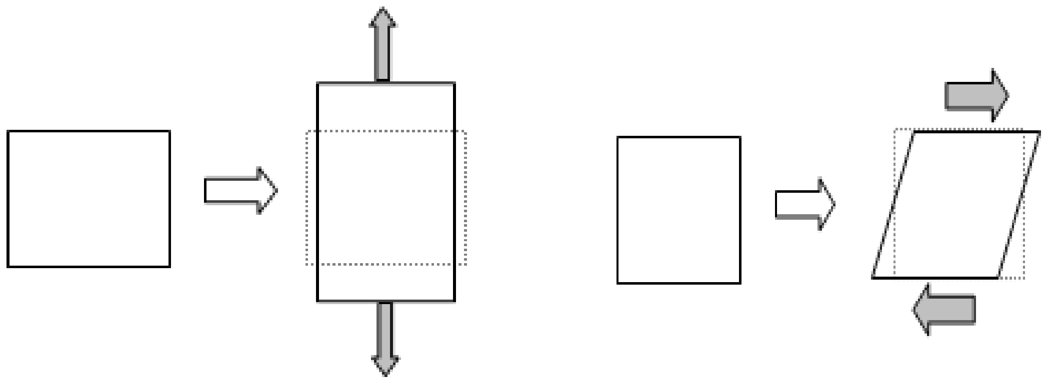
	In the direction of		
	x:	y:	z:
stress on the face normal to x:	σ_{xx}	σ_{xy}	σ_{xz}
stress on the face normal to y:	σ_{yx}	σ_{yy}	σ_{yz}
stress on the face normal to z:	σ_{zx}	σ_{zy}	σ_{zz}

Stress Tensor

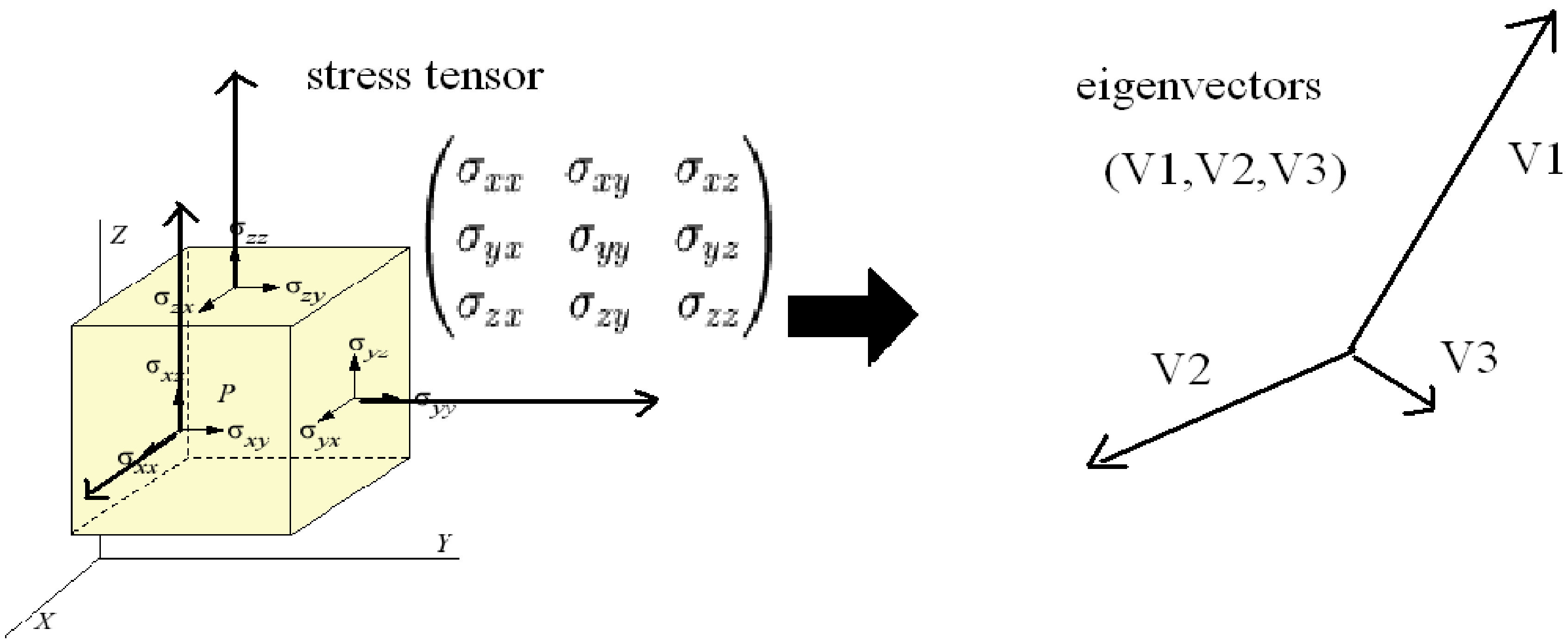
- A **'normal' stress is a stress perpendicular (i.e. normal)** to a specified surface
- A **shear stress acts tangentially** to the surface orientation
  - Stress tensor : characterised by **principle axes of tensor**
  - We can compose a 3x3 matrix called **Stress Tensor** representing the stress added to the box

- This is for computing how much the shape gets distorted by the stress



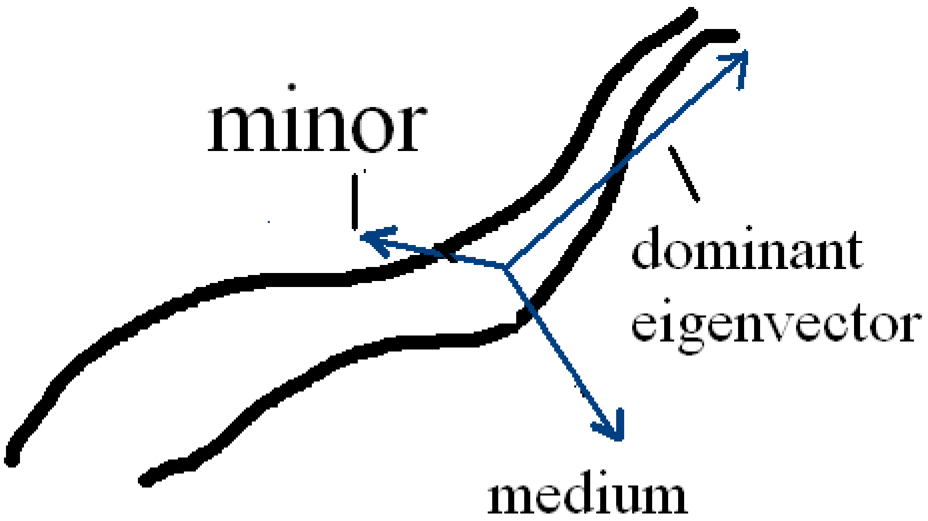
Computing Eigenvectors from the Stress Tensor

- 3x3 matrix results in **Eigenvalues** (scale) of normal stress along **eigenvectors** (direction)
- form 3D co-ordinate system (locally) with mutually perp. Axes
- ordering by eigenvector referred to as **major, medium and minor eigenvectors**



Diffusion Tensor-MRI : diffusion tensor

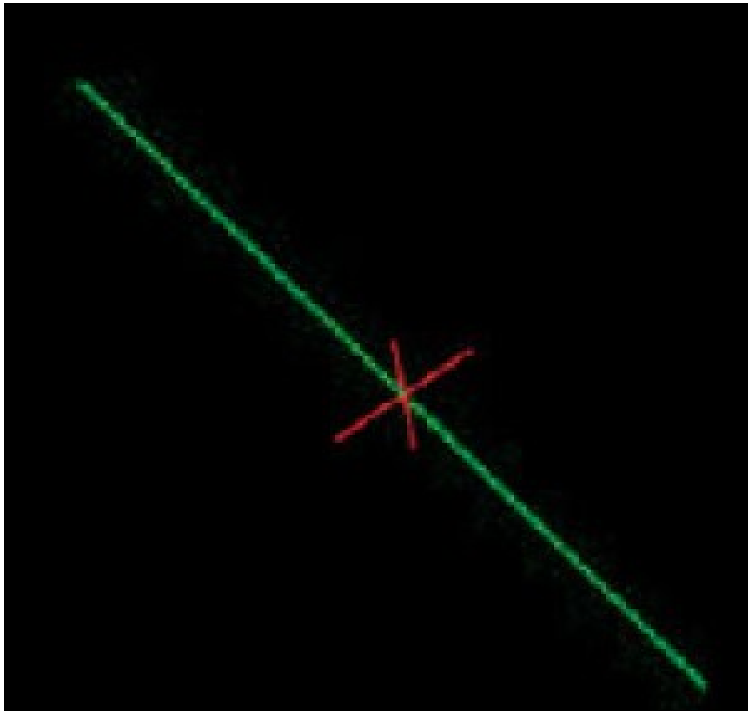
- Water molecules have **anisotropic diffusion in the body due to the cell shape and membrane properties**
  - **Neural fibers : long cylindrical cells filled with fluid**
  - **Water diffusion rate is fastest along the axis**
  - **Slowest in the two transverse directions**
  - **brain functional imaging** by detecting the anisotropy
  - Again, we can represent the diffusion tensor by the eigenvectors and eigenvalues

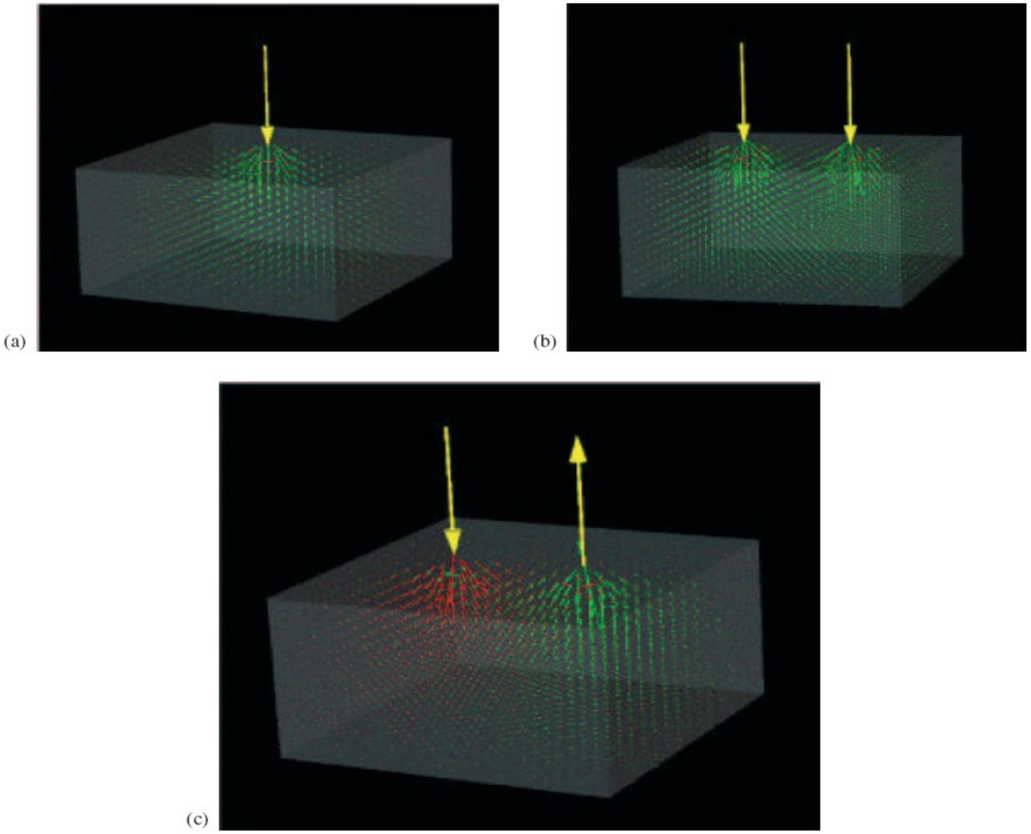


Tensors : Visualisation Methods

- For visualization of tensors, we have to visualize
  - three vectors orthogonal to each other
  - At every sample point in the 3D space
- **Vector methods**
  - **hedgehogs**
  - **Streamline, hyper-streamline**
- **Glyphs**
  - 3D ellipses particularly appropriate (3 modes of variation)

Hedgehogs





- Using hedgehogs to draw the three eigenvectors
  - The length is the stress value
- Good for simple cases as above
  - Applying forces to the box
  - Green represents positive, red negative

Hedgehogs

- Not good if
  - The grid is coarse
  - The stress is non-uniform, non-linear across the object

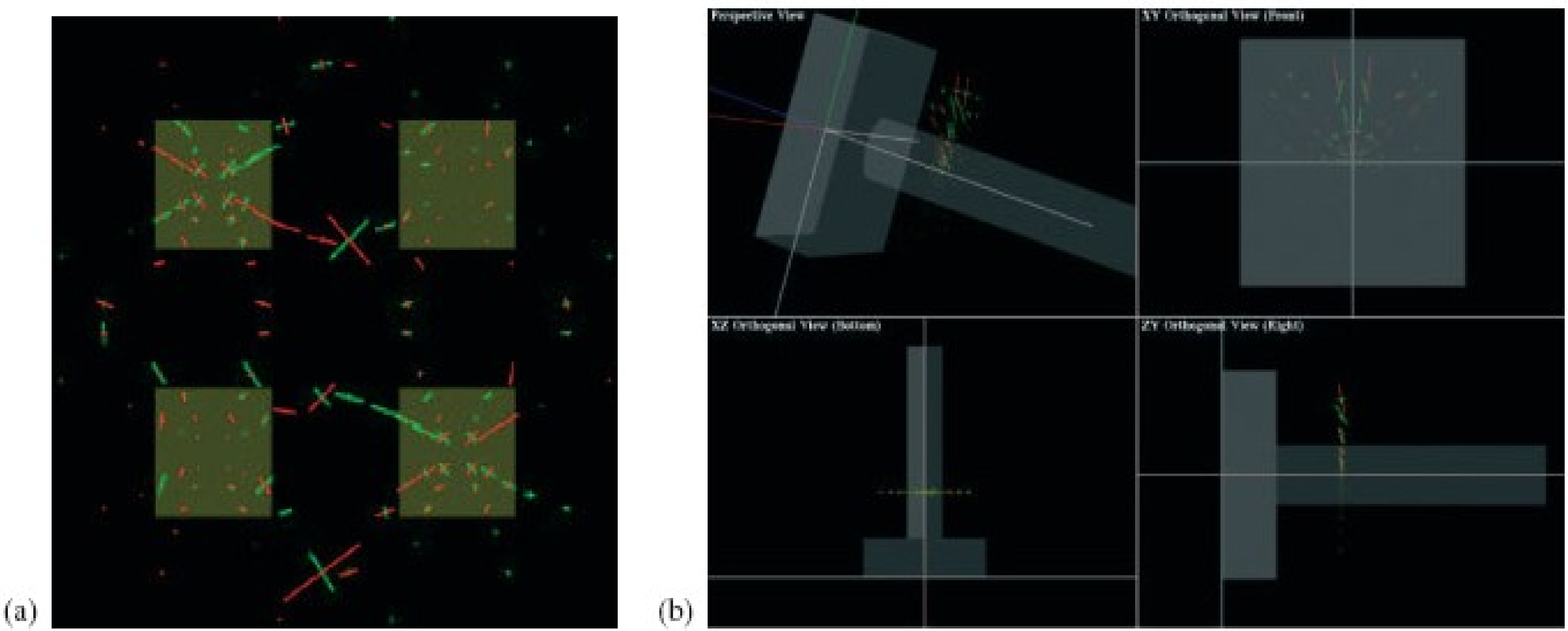
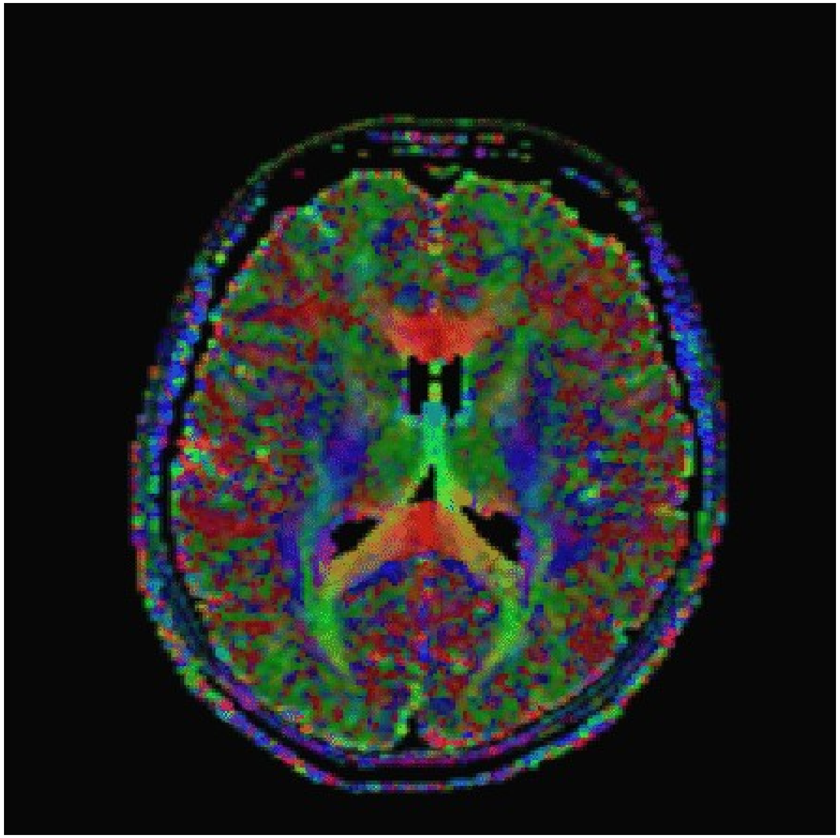


Figure 7. (a) Principal stress hedgehogs for one layer of four-pile group close to the surface, (b) principal stress hedgehogs for a layer just beneath the pile cap for a single pile.

Tensor Visualisation by Colormap



Source: R. Sierra

- Visualise just the major **eigenvectors as a vector field**
  - alternatively medium or minor eigenvector

e.g. Major eigenvector direction visualised with $(u,v,w) \longrightarrow (r,g,b)$ colourmap.

Streamlines for tensor visualisation

- **Each eigenvector defines a vector field**
- Using the eigenvector to create the streamline
  - **We can use the Major vector, the medium and the minor vector to generate different streamlines**

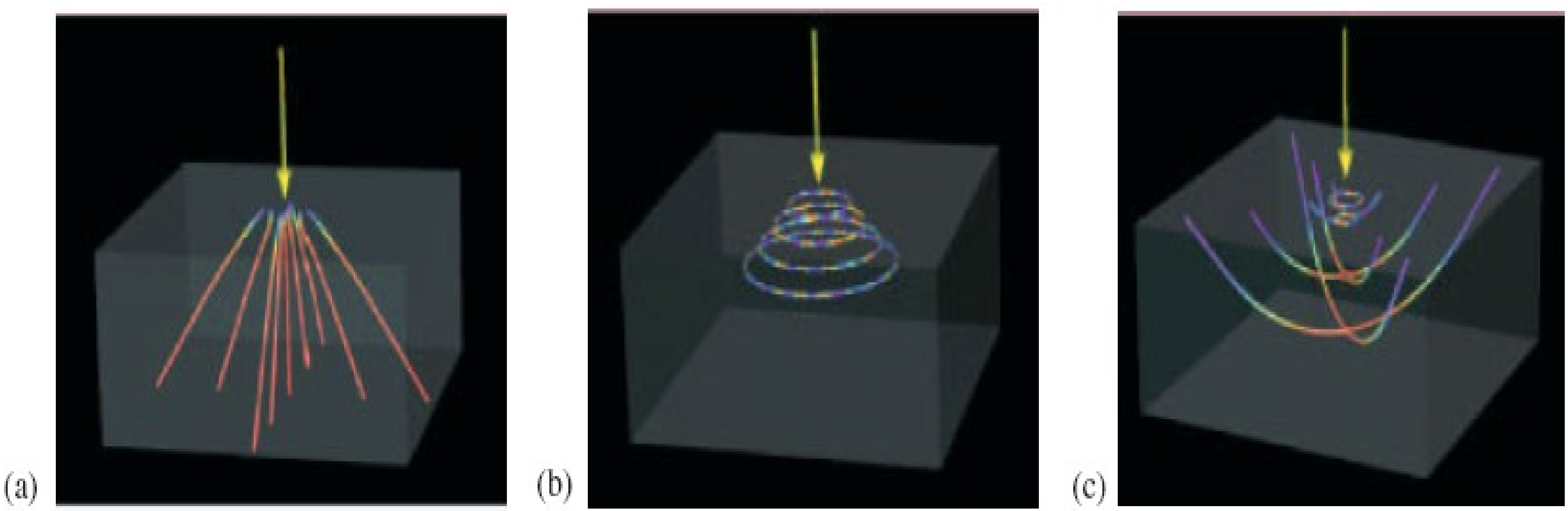


Figure 8. Hyperstreamlines for minor, intermediate and major principal stress for a point-load.

Streamlines for MRI

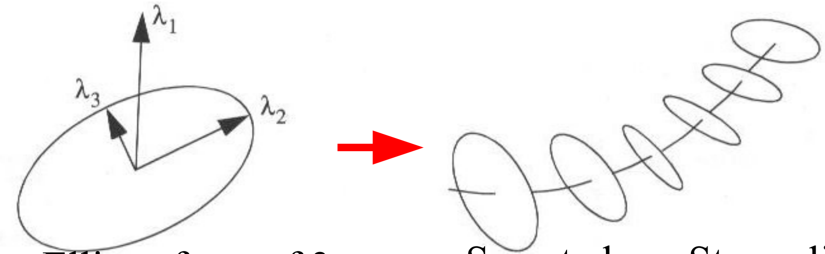
- **For DT-MRI, major vector** indicates nerve pathways or stress directions.
- Visualization of the brain nerves by the streamlines based on the major eigenvectors of the water diffusion

http://www.cmiv.liu.se/

Hyper-streamlines

[Delmarcelle et al. '93]

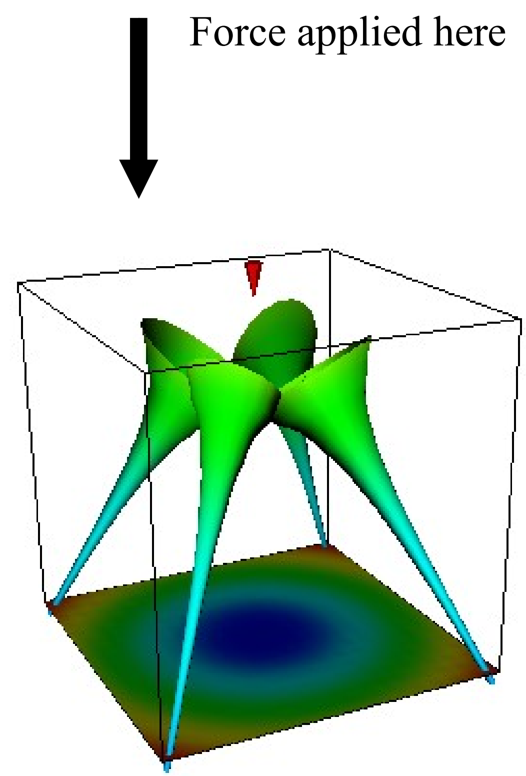
- Construct a **streamline from vector field of major eigenvector**



Ellipse from of 3 orthogonal eigenvectors

Swept along Streamline

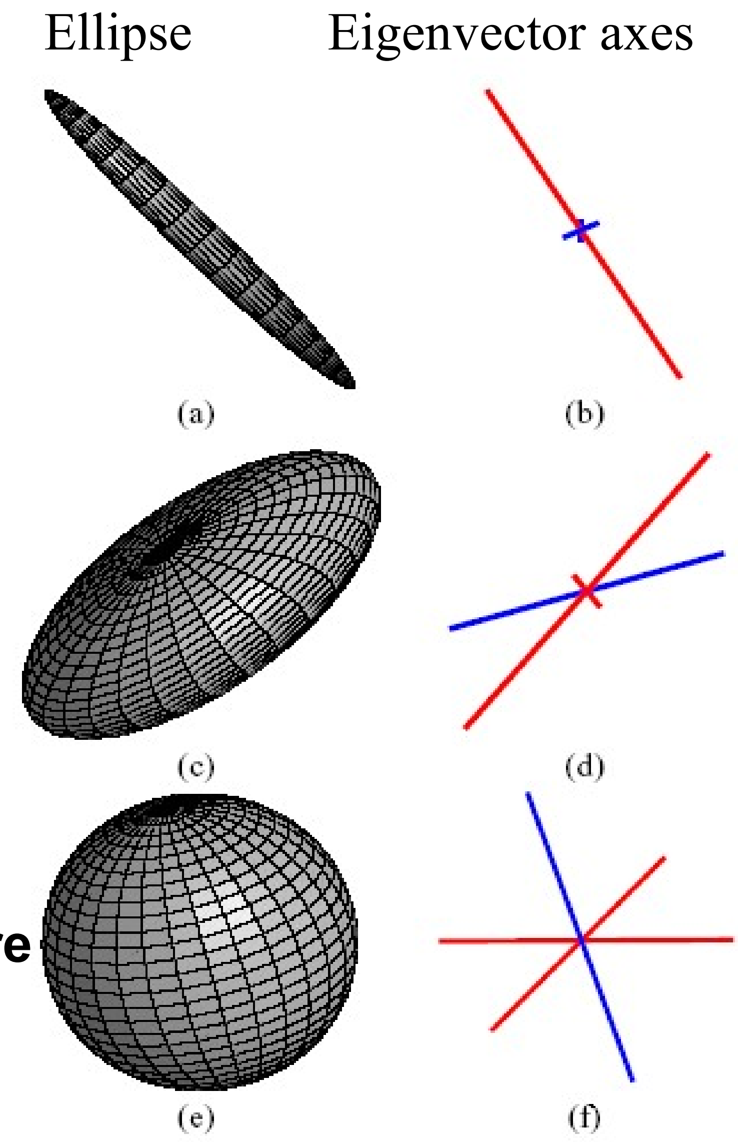
- **Form ellipse together with medium and minor eigenvector**
  - both are orthogonal to streamline direction
- **Sweep ellipse** along streamline
  - ***Hyper-Streamline*** ***(type of stream polygon)***



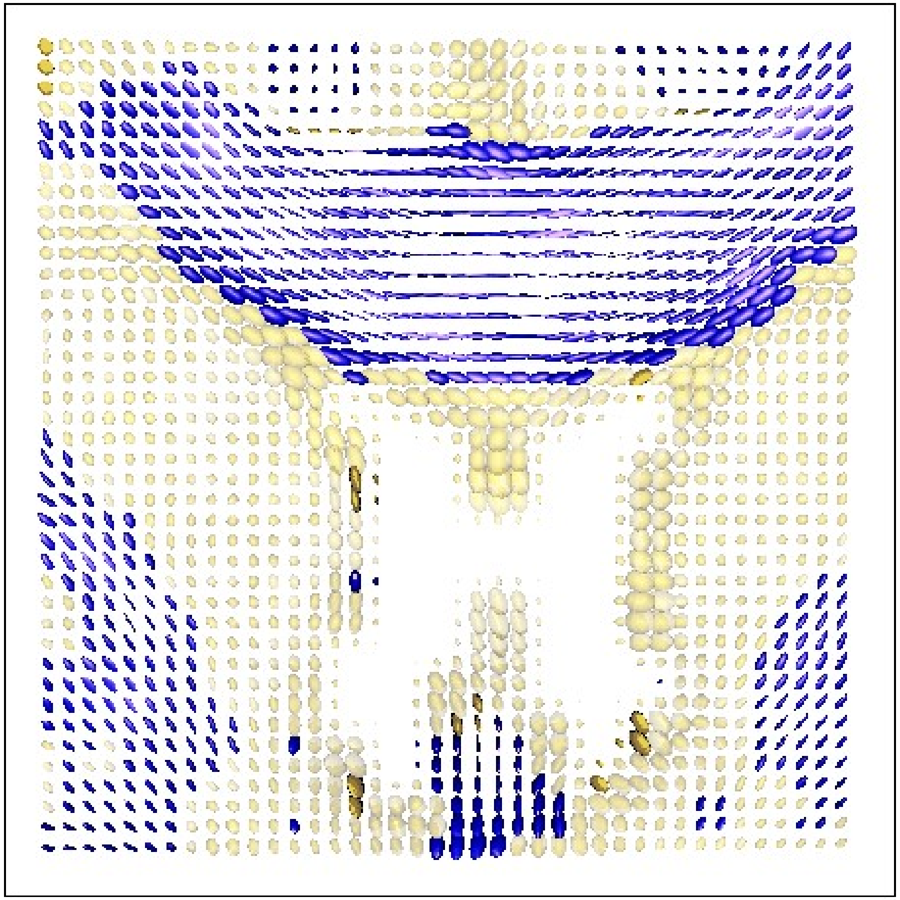
Visualizing the information of the three eigenvectors altogether !

Tensor Glyphs

- **Ellipses**
  - **rotated into coordinate system defined by eigenvectors** of tensor
  - axes are **scaled by the eigenvalues**
  - **very suitable as 3 modes of variation**
- **Classes of tensor:**
  - (a,b) - **large major eigenvalue**
    - ellipse approximates a **line**
  - (c,d) - **large major and medium eigenvalue**
    - ellipse approximates a **plane**
  - (e,f) - **all similar** - ellipse approximates a **sphere**



Diffusion Tensor Visualisation



Baby's brain image (source: R.Sierra)

The brain consists of different types of tissues with different diffusions

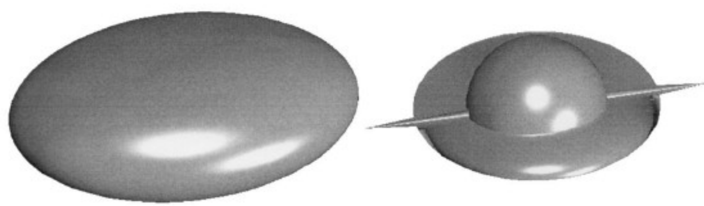
Anisotropic tensors indicate nerve pathway in brain:

- **Blue shape** – tensor approximates a line. (nerves)
- **Yellow shape** – tensor approximates a plane.
- **Yellow transparent shape** – ellipse approximates a sphere

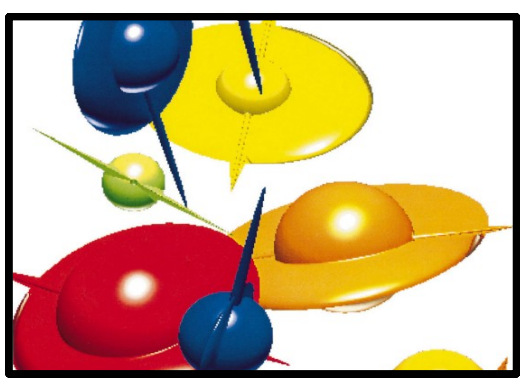
Colours needed due to **ambiguity in 3D shape**

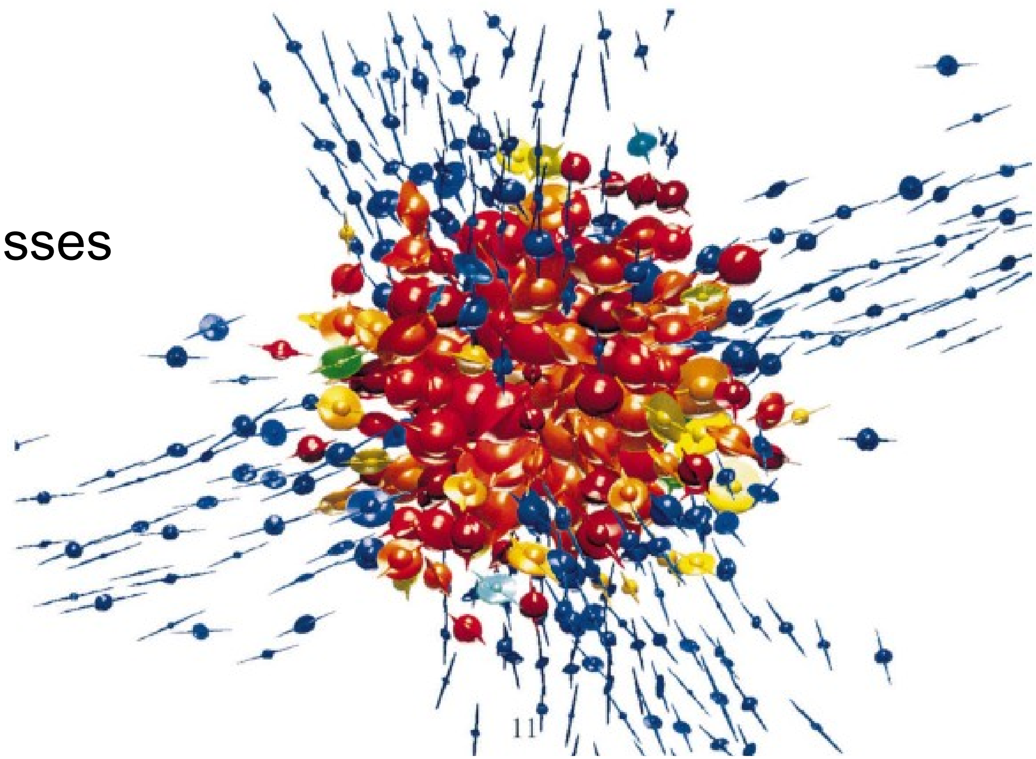
Example : tensor glyphs

- Glyphs with similar positional and modes of shape variation to ellipsoid used for **MRI diffusion tensor visualisation**
  - **disambiguates orientation**



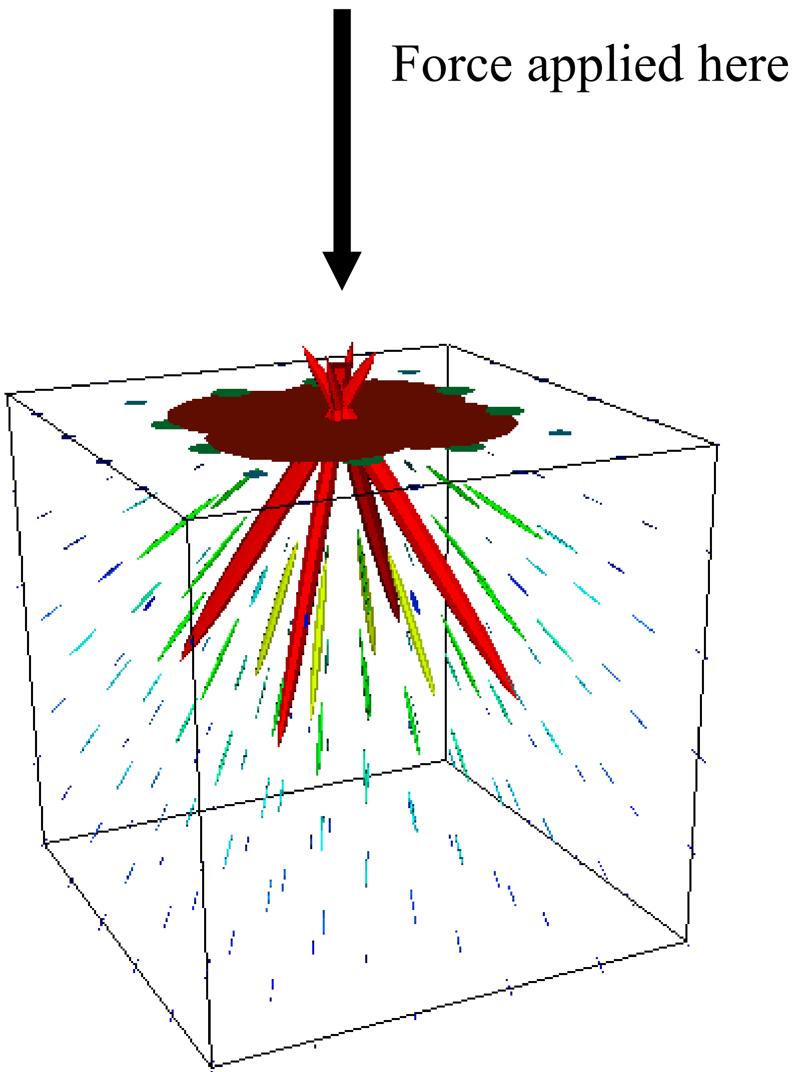
  - interpolated between classes





[Westin et al. '02]

Stress Ellipses

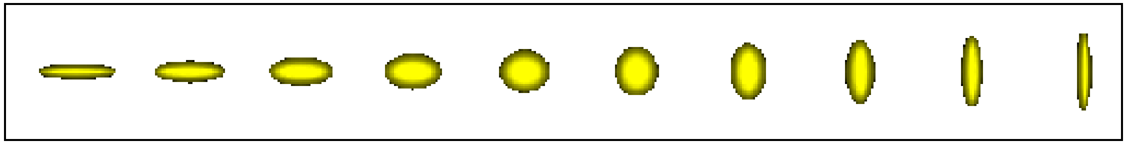


- Force applied to dense 3D solid – resulting **stress at 3D position in structure**
- Ellipses visualise the stress tensor
- Tensor Eigenvalues:
  - Large **major eigenvalue indicates principle direction of stress**
  - 'Temperature' **colourmap indicates size of major eigenvalue (magnitude of stress)**

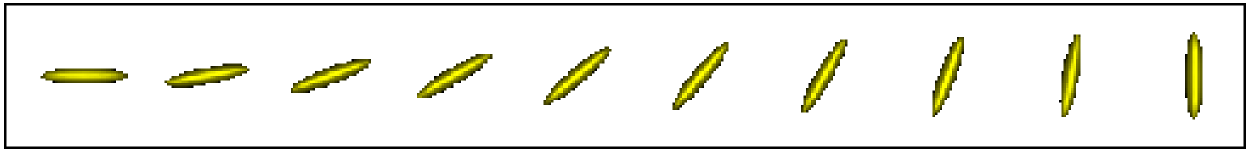
Interpolation of Tensors

- **How do we interpolate over tensors ?**

- Can **simply interpolate over eigen-components** individually:



- But if it **represents specific information** (e.g. nerve pathway) then **shape preserving methods are preferred**:



Summary

- **Tensor Visualisation**
  - **challenging**
  - for common rank 2 tensors in $\mathbb{R}^3$
    - common sources **stress / strain / MRI data**
  - a number of methods exist via **eigenanalysis decomposition of tensors**
    - **3D glyphs** – specifically **ellipsoids**
    - **vector and scalar field** methods
    - **hyper-streamlines**