The Lambda Calculus
The lambda-calculus

- If our previous language of arithmetic expressions was the simplest nontrivial programming language, then the lambda-calculus is the simplest *interesting* programming language...
  - Turing complete
  - higher order (functions as data)
- Indeed, in the lambda-calculus, *all* computation happens by means of function abstraction and application.
- The *e. coli* of programming language research
- The foundation of many real-world programming language designs (including ML, Haskell, Scheme, Lisp, ...)
Suppose we want to describe a function that adds three to any number we pass it. We might write

\[ \text{plus3 } x = \text{succ (succ (succ x))} \]

That is, “\text{plus3 } x \text{ is succ (succ (succ x))}.”
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That is, “\text{plus3 } x \text{ is succ (succ (succ x))}.”

Q: What is \text{plus3} itself?

A: \text{plus3} is the function that, given \( x \), yields \( \text{succ (succ (succ x))} \).

\[ \text{plus3} = \lambda x. \text{succ (succ (succ x))} \]

This function exists independent of the name \text{plus3}.

\( \lambda x. t \) is written “\text{fun } x \rightarrow t” in OCaml.
So \( \text{plus3} \ (\text{succ} \ 0) \) is just a convenient shorthand for “the function that, given \( x \), yields \( \text{succ} \ (\text{succ} \ (\text{succ} \ x)) \), applied to \( \text{succ} \ 0 \).”

\[
\text{plus3} \ (\text{succ} \ 0) = (\lambda x. \ \text{succ} \ (\text{succ} \ (\text{succ} \ x))) \ (\text{succ} \ 0)
\]
Consider the \( \lambda \)-abstraction

\[
g = \lambda f. f (f \ (\text{succ } 0))
\]

Note that the parameter variable \( f \) is used in the \textit{function} position in the body of \( g \). Terms like \( g \) are called \textit{higher-order} functions. If we apply \( g \) to an argument like \texttt{plus3}, the “substitution rule” yields a nontrivial computation:

\[
g \ \texttt{plus3} \\
= \ (\lambda f. f \ (f \ (\text{succ } 0))) \ (\lambda x. \text{succ} \ (\text{succ} \ (\text{succ} \ x))) \\
i.e. \ (\lambda x. \text{succ} \ (\text{succ} \ (\text{succ} \ x))) \\
\hspace{1cm} ((\lambda x. \text{succ} \ (\text{succ} \ (\text{succ} \ x))) \ (\text{succ } 0)) \\
i.e. \ (\lambda x. \text{succ} \ (\text{succ} \ (\text{succ} \ x))) \\
\hspace{1cm} (\text{succ} \ (\text{succ} \ (\text{succ} \ (\text{succ } 0)))) \\
i.e. \ \text{succ} \ (\text{succ} \ (\text{succ} \ (\text{succ} \ (\text{succ} \ (\text{succ } 0))))))
Abstractions Returning Functions

Consider the following variant of \( g \):

\[
\text{double} \quad = \quad \lambda f. \lambda y. f (f \ y)
\]

I.e., \texttt{double} is the function that, when applied to a function \( f \), yields a \textit{function} that, when applied to an argument \( y \), yields \( f (f \ y) \).
Example

double plus3 0
= (\lambda f. \lambda y. f (f y))
   (\lambda x. succ (succ (succ x)))
   0
i.e. (\lambda y. (\lambda x. succ (succ (succ x))))
   ((\lambda x. succ (succ (succ x))) y)
   0
i.e. (\lambda x. succ (succ (succ x)))
   ((\lambda x. succ (succ (succ x))) 0)
i.e. (\lambda x. succ (succ (succ x)))
   (succ (succ (succ 0)))
i.e. succ (succ (succ (succ (succ (succ 0)))))
The Pure Lambda-Calculus

As the preceding examples suggest, once we have $\lambda$-abstraction and application, we can throw away all the other language primitives and still have left a rich and powerful programming language.

In this language — the “pure lambda-calculus” — *everything* is a function.

- Variables always denote functions
- Functions always take other functions as parameters
- The result of a function is always a function
Formalities
Syntax

\[ t ::= \begin{array}{ll}
  \text{terms} \\
  x & \text{variable} \\
  \lambda x. t & \text{abstraction} \\
  t \ t & \text{application}
\end{array} \]

Terminology:

- terms in the pure \( \lambda \)-calculus are often called \( \lambda \)-terms
- terms of the form \( \lambda x. \ t \) are called \( \lambda \)-abstractions or just abstractions
Syntactic conventions

Since λ-calculus provides only one-argument functions, all multi-argument functions must be written in curried style.

The following conventions make the linear forms of terms easier to read and write:

- Application associates to the left
  
  \[ E.g., t \ u \ v \ means \ (t \ u) \ v, \ not \ t \ (u \ v) \]

- Bodies of λ- abstractions extend as far to the right as possible
  
  \[ E.g., \ \lambda x. \ \lambda y. \ x \ y \ means \ \lambda x. \ (\lambda y. \ x \ y), \ not \ \lambda x. \ (\lambda y. \ x) \ y \]
Scope

The $\lambda$-abstraction term $\lambda x.t$ binds the variable $x$. The scope of this binding is the body $t$. Occurrences of $x$ inside $t$ are said to be \emph{bound} by the abstraction. Occurrences of $x$ that are \emph{not} within the scope of an abstraction binding $x$ are said to be \emph{free}.

\[
\lambda x. \, \lambda y. \, x \, y \, z
\]
The $\lambda$-abstraction term $\lambda x. t$ binds the variable $x$. The scope of this binding is the body $t$. Occurrences of $x$ inside $t$ are said to be bound by the abstraction. Occurrences of $x$ that are not within the scope of an abstraction binding $x$ are said to be free.

\[
\lambda x. \lambda y. x \ y \ z \\
\lambda x. (\lambda y. z \ y) \ y
\]
Values

\[ v ::= \lambda x.t \]

values

abstraction value
Operational Semantics

Computation rule:

\[ (\lambda x. t_{12}) v_2 \rightarrow [x \mapsto v_2] t_{12} \quad (E\text{-AppAbs}) \]

Notation: \([x \mapsto v_2] t_{12}\) is “the term that results from substituting free occurrences of \(x\) in \(t_{12}\) with \(v_{12}\).”
Operational Semantics

Computation rule:

\[(\lambda x. t_{12}) v_2 \rightarrow [x \mapsto v_2] t_{12}\]  \hspace{1cm} (E-AppAbs)

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Congruence rules:

\[
\frac{t_1 \rightarrow t_1'}{t_1 \cdot t_2 \rightarrow t_1' \cdot t_2} \hspace{1cm} (E-App1)
\]

\[
\frac{t_2 \rightarrow t_2'}{v_1 \cdot t_2 \rightarrow v_1 \cdot t_2'} \hspace{1cm} (E-App2)
\]
Terminology

A term of the form \((\lambda x. t) v\) — that is, a \(\lambda\)-abstraction applied to a \textit{value} — is called a \textit{redex} (short for “reducible expression”).
Alternative evaluation strategies

Strictly speaking, the language we have defined is called the *pure, call-by-value lambda-calculus*. The evaluation strategy we have chosen — *call by value* — reflects standard conventions found in most mainstream languages. Some other common ones:

- Call by name (cf. Haskell)
- Normal order (leftmost/outermost)
- Full (non-deterministic) beta-reduction
Programming in the Lambda-Calculus
Multiple arguments

Above, we wrote a function `double` that returns a function as an argument.

\[
double = \lambda f. \lambda y. f (f \, y)
\]

This idiom — a \(\lambda\)-abstraction that does nothing but immediately yield another abstraction — is very common in the \(\lambda\)-calculus. In general, \(\lambda x. \lambda y. t\) is a function that, given a value \(v\) for \(x\), yields a function that, given a value \(u\) for \(y\), yields \(t\) with \(v\) in place of \(x\) and \(u\) in place of \(y\). That is, \(\lambda x. \lambda y. t\) is a two-argument function.

(Recall the discussion of *currying* in OCaml.)
Syntactic conventions

Since $\lambda$-calculus provides only one-argument functions, all multi-argument functions must be written in curried style. The following conventions make the linear forms of terms easier to read and write:

- Application associates to the left
  
  \[ E.g., \ t \ u \ v \ means \ (t \ u) \ v, \ not \ t \ (u \ v) \]

- Bodies of $\lambda$- abstractions extend as far to the right as possible
  
  \[ E.g., \ \lambda x. \ \lambda y. \ x \ y \ means \ \lambda x. \ (\lambda y. \ x \ y), \ not \ \lambda x. \ (\lambda y. \ x) \ y \]
The “Church Booleans”

\[
\text{tru} = \lambda t. \lambda f. t \\
\text{fls} = \lambda t. \lambda f. f
\]

\[
\text{tru} \ v \ w \\
= (\lambda t. \lambda f. t) \ v \ w \quad \text{by definition} \\
\rightarrow (\lambda f. v) \ w \quad \text{reducing the underlined redex} \\
\rightarrow v \quad \text{reducing the underlined redex}
\]

\[
\text{fls} \ v \ w \\
= (\lambda t. \lambda f. f) \ v \ w \quad \text{by definition} \\
\rightarrow (\lambda f. f) \ w \quad \text{reducing the underlined redex} \\
\rightarrow w \quad \text{reducing the underlined redex}
\]
Functions on Booleans

\[ \text{not} = \lambda b. \text{fls} \text{ if } v \text{ is } \text{tru} \text{ and } \text{tru} \text{ if } v \text{ is } \text{fls}. \]
Functions on Booleans

\[ \text{and} = \lambda b. \lambda c. b \ c \ \text{fls} \]

That is, \text{and} is a function that, given two boolean values \( v \) and \( w \), returns \( w \) if \( v \) is \text{tru} and \text{fls} if \( v \) is \text{fls}.

Thus \text{and} \( v \ w \) yields \text{tru} if both \( v \) and \( w \) are \text{tru} and \text{fls} if either \( v \) or \( w \) is \text{fls}.
Pairs

pair = $\lambda f. \lambda s. \lambda b. \; b \; f \; s$
fst = $\lambda p. \; p \; \text{tru}$
snd = $\lambda p. \; p \; \text{fls}$

That is, \text{pair} \; v \; w is a function that, when applied to a boolean value \( b \), applies \( b \) to \( v \) and \( w \).
By the definition of booleans, this application yields \( v \) if \( b \) is \text{tru} and \( w \) if \( b \) is \text{fls}, so the first and second projection functions \text{fst} and \text{snd} can be implemented simply by supplying the appropriate boolean.
Example

\[
\text{fst (pair } v \ w) = \text{fst } ((\lambda f. \lambda s. \lambda b. b \ f \ s) \ v \ w) \quad \text{by definition}
\]
\[
\rightarrow \text{fst } ((\lambda s. \lambda b. b \ v \ s) \ w)
\]
\[
\rightarrow \text{fst } (\lambda b. b \ v \ w)
\]
\[
= (\lambda p. p \ \text{tru}) (\lambda b. b \ v \ w) \quad \text{by definition}
\]
\[
\rightarrow (\lambda b. b \ v \ w) \ \text{tru}
\]
\[
\rightarrow \text{tru } v \ w
\]
\[
\rightarrow^* v \quad \text{as before.}
\]
Church numerals

Idea: represent the number $n$ by a function that “repeats some action $n$ times.”

\[
\begin{align*}
c_0 &= \lambda s. \lambda z. z \\
c_1 &= \lambda s. \lambda z. s \ z \\
c_2 &= \lambda s. \lambda z. s\ (s\ z) \\
c_3 &= \lambda s. \lambda z. s\ (s\ (s\ z))
\end{align*}
\]

That is, each number $n$ is represented by a term $c_n$ that takes two arguments, $s$ and $z$ (for “successor” and “zero”), and applies $s$, $n$ times, to $z$. 
Successor:
Functions on Church Numerals

Successor:

\[ \text{succ} = \lambda n. \lambda s. \lambda z. s \ (n \ s \ z) \]
Functions on Church Numerals

Successor:

\[ scc = \lambda n. \lambda s. \lambda z. s (n s z) \]

Addition:
Functions on Church Numerals

Successor:

\[ \text{sc}c = \lambda n. \lambda s. \lambda z. s (n s z) \]

Addition:

\[ \text{pl}us = \lambda m. \lambda n. \lambda s. \lambda z. m s (n s z) \]
Functions on Church Numerals

Successor:

\[ \text{sc} = \lambda n. \lambda s. \lambda z. s (n s z) \]

Addition:

\[ \text{plus} = \lambda m. \lambda n. \lambda s. \lambda z. m s (n s z) \]

Multiplication:

What about predecessor?
Functions on Church Numerals

Successor:

\[
\text{succ} = \lambda n. \lambda s. \lambda z. s \ (n \ s \ z)
\]

Addition:

\[
\text{plus} = \lambda m. \lambda n. \lambda s. \lambda z. m \ s \ (n \ s \ z)
\]

Multiplication:

\[
\text{times} = \lambda m. \lambda n. m \ (\text{plus} \ n) \ c_0
\]
Functions on Church Numerals

Successor:

\[ \text{succ} = \lambda n. \lambda s. \lambda z. s \ (n \ s \ z) \]

Addition:

\[ \text{plus} = \lambda m. \lambda n. \lambda s. \lambda z. m \ s \ (n \ s \ z) \]

Multiplication:

\[ \text{times} = \lambda m. \lambda n. m \ (\text{plus} \ n) \ \text{c}_0 \]

Zero test:
Functions on Church Numerals

Successor:
\[ \text{sc} = \lambda n. \lambda s. \lambda z. s (n s z) \]

Addition:
\[ \text{plus} = \lambda m. \lambda n. \lambda s. \lambda z. m s (n s z) \]

Multiplication:
\[ \text{times} = \lambda m. \lambda n. m (\text{plus} n) \, c_0 \]

Zero test:
\[ \text{iszro} = \lambda m. m (\lambda x. \text{fls}) \, \text{tru} \]
Functions on Church Numerals

Successor:

\[ sc\,c = \lambda n. \lambda s. \lambda z. s\,(n\,s\,z) \]

Addition:

\[ p\,l\,u\,s\, = \lambda m. \lambda n. \lambda s. \lambda z. m\,s\,(n\,s\,z) \]

Multiplication:

\[ t\,i\,m\,e\,s\, = \lambda m. \lambda n. m\,(p\,l\,u\,s\,\,n)\,c_0 \]

Zero test:

\[ i\,s\,z\,r\,o\, = \lambda m. m\,(\lambda x. f\,l\,s)\,t\,r\,u \]

What about predecessor?
Predecessor

\[ \text{zz} = \text{pair } c_0 \ c_0 \]

\[ \text{ss} = \lambda p. \ \text{pair } (\text{snd } p) \ (\text{scc } (\text{snd } p)) \]

\[ \text{prd} = \lambda m. \ \text{fst } (m \ \text{ss } \text{zz}) \]
Normal forms

Recall:

- A *normal form* is a term that cannot take an evaluation step.
- A *stuck* term is a normal form that is not a value.

Are there any stuck terms in the pure $\lambda$-calculus? Prove it.
Normal forms

Recall:

- A *normal form* is a term that cannot take an evaluation step.
- A *stuck* term is a normal form that is not a value.

Are there any stuck terms in the pure $\lambda$-calculus? Prove it.
Does every term evaluate to a normal form? Prove it.
Divergence

\[
\text{omega} \quad = \quad (\lambda x. \, x \, x) \, (\lambda x. \, x \, x)
\]

Note that omega evaluates in one step to itself!
So evaluation of omega never reaches a normal form: it diverges.
Divergence

\[
\omega = (\lambda x. \ x \ x) \ (\lambda x. \ x \ x)
\]

Note that \textit{\texttt{omega}} evaluates in one step to itself!
So evaluation of \textit{\texttt{omega}} never reaches a normal form: it \textit{diverges}.

Being able to write a divergent computation does not seem very useful in itself. However, there are variants of \textit{\texttt{omega}} that are very useful...
Recursion in the Lambda-Calculus
Iterated Application

Suppose $f$ is some $\lambda$-abstraction, and consider the following term:

$$Y_f = (\lambda x. f(x x)) (\lambda x. f(x x))$$
Iterated Application

Suppose \( f \) is some \( \lambda \)-abstraction, and consider the following term:

\[
Y_f = (\lambda x. f (x x)) (\lambda x. f (x x))
\]

Now the “pattern of divergence” becomes more interesting:

\[
egin{align*}
Y_f &= (\lambda x. f (x x)) (\lambda x. f (x x)) \\
&\quad \rightarrow f ((\lambda x. f (x x)) (\lambda x. f (x x))) \\
&\quad \rightarrow f (f ((\lambda x. f (x x)) (\lambda x. f (x x)))) \\
&\quad \rightarrow f (f (f ((\lambda x. f (x x)) (\lambda x. f (x x))))) \\
&\quad \rightarrow \ldots
\end{align*}
\]
$Y_f$ is still not very useful, since (like \textit{omega}), all it does is diverge. Is there any way we could “slow it down”? 
Delaying divergence

\[
\text{poisonpill} = \lambda y. \text{omega}
\]

Note that \textit{poisonpill} is a value — it will only diverge when we actually apply it to an argument. This means that we can safely pass it as an argument to other functions, return it as a result from functions, etc.

\[
(\lambda p. \text{fst (pair p fls) tru}) \text{poisonpill}
\]

\[
\rightarrow
\]

\[
\text{fst (pair poisonpill fls) tru}
\]

\[
\rightarrow^*
\]

\[
\text{poisonpill tru}
\]

\[
\rightarrow
\]

\[
\text{omega}
\]

\[
\rightarrow
\]

\[
\ldots
\]

\text{Cf. thunks in OCaml.}
A delayed variant of omega

Here is a variant of omega in which the delay and divergence are a bit more tightly intertwined:

\[
\text{omegav} = \\
\lambda y. (\lambda x. (\lambda y. x x y)) (\lambda x. (\lambda y. x x y)) y
\]

Note that omegav is a normal form. However, if we apply it to any argument v, it diverges:

\[
\text{omegav} v \\
= \\
(\lambda y. (\lambda x. (\lambda y. x x y)) (\lambda x. (\lambda y. x x y)) y) v \\
\quad \rightarrow \\
(\lambda x. (\lambda y. x x y)) (\lambda x. (\lambda y. x x y)) v \\
\quad \rightarrow \\
(\lambda y. (\lambda x. (\lambda y. x x y)) (\lambda x. (\lambda y. x x y)) y) v \\
= \\
\text{omegav} v
\]
Another delayed variant

Suppose \( f \) is a function. Define

\[
Z_f = \lambda y. (\lambda x. f (\lambda y. x x y)) (\lambda x. f (\lambda y. x x y)) y
\]

This term combines the “added \( f \)” from \( Y_f \) with the “delayed divergence” of \( \text{omegav} \).
If we now apply $Z_f$ to an argument $v$, something interesting happens:

$$Z_f \ v$$

$$= (\lambda y. (\lambda x. f (\lambda y. x \ x \ y)) \ (\lambda x. f (\lambda y. x \ x \ y)) \ y) \ v$$

$$\longrightarrow (\lambda x. f (\lambda y. x \ x \ y)) \ (\lambda x. f (\lambda y. x \ x \ y)) \ v$$

$$\longrightarrow f (\lambda y. (\lambda x. f (\lambda y. x \ x \ y)) \ (\lambda x. f (\lambda y. x \ x \ y)) \ y) \ v$$

$$= f \ Z_f \ v$$

Since $Z_f$ and $v$ are both values, the next computation step will be the reduction of $f \ Z_f$ — that is, before we “diverge,” $f$ gets to do some computation.

Now we are getting somewhere.
Recursion

Let

\[ f = \lambda fct. \lambda n. \]
\[ \quad \text{if } n=0 \text{ then } 1 \]
\[ \quad \text{else } n \times (fct (\text{pred } n)) \]

\( f \) looks just the ordinary factorial function, except that, in place of a recursive call in the last time, it calls the function \( fct \), which is passed as a parameter.

N.b.: for brevity, this example uses “real” numbers and booleans, infix syntax, etc. It can easily be translated into the pure lambda-calculus (using Church numerals, etc.).
We can use $Z$ to “tie the knot” in the definition of $f$ and obtain a real recursive factorial function:

$$Z_f 3 \rightarrow^{*}$$

$$f \ Z_f \ 3$$

$$= (\lambda fct. \ \lambda n. \ ...) \ Z_f \ 3 \rightarrow \rightarrow$$

$$\text{if } 3=0 \text{ then } 1 \text{ else } 3 \ast (Z_f \ (\text{pred } 3)) \rightarrow^{*}$$

$$3 \ast (Z_f \ (\text{pred } 3))) \rightarrow$$

$$3 \ast (Z_f \ 2) \rightarrow^{*}$$

$$3 \ast (f \ Z_f \ 2)$$

...
A Generic \( Z \)

If we define

\[
Z = \lambda f. \ Z_f
\]

i.e.,

\[
Z = \lambda f. \ \lambda y. \ (\lambda x. \ f \ (\lambda y. \ x \ x \ y)) \ (\lambda x. \ f \ (\lambda y. \ x \ x \ y)) \ y
\]

then we can obtain the behavior of \( Z_f \) for any \( f \) we like, simply by applying \( Z \) to \( f \).

\[
Z \ f \rightarrow Z_f
\]
For example:

\[
\text{fact} \quad = \quad Z \left( \lambda fct. \right. \\
\quad \quad \left. \lambda n. \right. \\
\quad \quad \quad \begin{array}{l}
\text{if } n=0 \text{ then } 1 \\
\text{else } n \times (fct \ (\text{pred } n))
\end{array}
\]
The term $Z$ here is essentially the same as the $\text{fix}$ discussed the book.

$$Z = \lambda f. \lambda y. (\lambda x. f (\lambda y. x x y)) (\lambda x. f (\lambda y. x x y)) \; y$$

$$\text{fix} = \lambda f. (\lambda x. f (\lambda y. x x y)) (\lambda x. f (\lambda y. x x y))$$

$Z$ is hopefully slightly easier to understand, since it has the property that $Z \; f \; v \rightarrow^* f \; (Z \; f) \; v$, which $\text{fix}$ does not (quite) share.