Random Graphs

Social and Technological Networks

Rik Sarkar

University of Edinburgh, 2019.
Today

- What are random graphs?
  - How can we define “random graphs”?
- Some properties of random graphs
Erdos – Renyi Random graphs
Erdos – Renyi Random graphs

$\mathcal{G}(n, p)$

- $n$: number of vertices
- $p$: probability that any particular edge exists

- Take $V$ with $n$ vertices
- Consider each possible edge. Add it to $E$ with probability $p$
Expected number of edges in an ER graph

• Expected total number of edges

• Expected number of edges at any vertex
Expected number of edges

• Expected total number of edges \( \binom{n}{2} p \)

• Expected number of edges at any vertex \( (n - 1)p \)
Expected number of edges

• For \( p = \frac{c}{n - 1} \)

• The expected degree of a node is : ?
Isolated vertices

• How likely is it that the graph has isolated vertices?
Isolated vertices

- How likely is it that the graph has isolated vertices?

- What happens to the number of isolated vertices as $p$ increases?
Threshold phenomenon: Probability or number of isolated vertices

- The tipping point, phase transition

- Common in many real systems
Probability of Isolated vertices

• Isolated vertices are

• Likely when: \( p < \frac{\ln n}{n} \)

• Unlikely when: \( p > \frac{\ln n}{n} \)

• Let’s deduce
Terminology of high probability

• Poly(n) means a polynomial in n
• A polynomial in n is considered reasonably ‘large’
  – Whereas something like constant, or log n is considered ‘small’

• Something happens with high probability if

\[
\Pr[\text{event}] \geq \left(1 - \frac{1}{\text{poly}(n)}\right)
\]

• Thus for large n, w.h.p there is no isolated vertex
• Expected number of isolated vertices is miniscule
Useful inequalities

\[
\left(1 + \frac{1}{x}\right)^x \leq e
\]

\[
\left(1 - \frac{1}{x}\right)^x \leq \frac{1}{e}
\]
Union bound

• For events A, B, C ...

• $\Pr[A \text{ or } B \text{ or } C \ldots] \leq \Pr[A] + \Pr[B] + \Pr[C] + \ldots$
• Theorem 1:
• If \[ p = (1 + \epsilon) \frac{\ln n}{n - 1} \]

• Then the probability that there exists an isolated vertex
\[ \leq \frac{1}{n^\epsilon} \]
• Theorem 2

• For \[ p = (1 - \epsilon) \frac{\ln n}{n - 1} \]

• Probability that vertex v is isolated \[ \geq \frac{1}{(2n)^{1-\epsilon}} \]
• **Theorem 2**
• For \( p = (1 - \epsilon) \frac{\ln n}{n - 1} \)

• Probability that vertex v is isolated \( \geq \frac{1}{(2n)^{1-\epsilon}} \)

• Expected number of isolated vertices:
\[
\geq \frac{n}{(2n)^{1-\epsilon}} = \frac{n^\epsilon}{2}
\]

Polynomial in n
Global CC in ER graphs

• What happens when $p$ is very small (almost 0)?

• What happens when $p$ is very large (almost 1)?
Global CC in ER graphs

- What happens at the tipping point?
Theorem

• For $p = c \frac{\ln n}{n}$

• Global cc in ER graphs is vanishingly small

$$\lim_{n \to \infty} cc(G) = \lim_{n \to \infty} \frac{\# \text{ closed triads}}{\# \text{ all triads}} = 0$$
• In other words, there is no constant $c$
  – Such that $cc(\text{ER-graph}) > c$
  – At the tipping point
Random graphs: Emergence of giant component

- Suppose $N_G$ is the size of the largest connected component in an ER graph

- How does $N_G/N$ change with $p$?

- When is $N_G/N$ at least a constant?
  - (giant component: at least a constant fraction of nodes)
Giant component

- When $p = (1-\varepsilon)/n$
  - W.h.p no GC, components of size $O(\log n)$

- When $p = (1+\varepsilon)/n$
  - W.h.p GC exists, where $N_G/N \sim \varepsilon$

- When $p = 1/n$
  - W.h.p Largest component has size $n^{2/3}$
Project

- Topics will be out today.
- Check email/piazza

- You can work/discuss in groups 2 of 3
  - But final submission is individual.

- Select Project by Friday Oct 11
  - Don’t rush. Think carefully what you might do in that topic. Look at the dataset.
  - Once selected. Don’t delay. Start working immediately.

- Final submission due: Thursday November 14.
Project

• October 21:
  – Due date for midway project plan.

• Submit a 1 page description of topic and problem statement, and how you plan to approach it. And get comments.

• No marks. Feedback only.

• Office hours: Tuesdays 2pm. IF 3.45.