Power law networks

Social and Technological Networks

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Degree distribution

• A way of characterizing networks
• More complex than single numbers
• Many standard networks are known to have “standard” degree distributions
• Gives ways to incorporate notions of “popularity” and understand them
Degree distribution

• As a function of $k$, what fraction of pages in the network have $k$ links?
• A histogram

• What does it look like in a random graph?
Degree distribution of a random graph

• Probability that a node has degree \( k \) is:
  – Given by binomial distribution:

\[
\binom{n-1}{k} p^k (1-p)^{n-1-k}
\]

- Possible sets of \( k \) edges
- Probability that all \( k \) are chosen
- Probability that others are not chosen
Degree distribution in a random graph

- Probabilities fall off really fast away from the peak
  - Exponentially fast with $k$
  - Very low and high degree are very very unlikely
Degree distribution in www

• Suppose we take a real network like the world wide web, and compute degree distribution. What does that look like?
• Let’s try.
Degree distribution in www

• For www snapshots, degree distribution follows approximately $\frac{1}{k^2}$
Power law networks

• With degree distribution \( \frac{1}{k^\alpha} \)

• For some constant \( \alpha \)
What do power law networks mean

• Most nodes have a low degree
• There are several hubs with high degree
  – Heavy tail
  – Probability drops polynomially
  • Slower than exponentially
Hubs in power law networks

- Highly connected people/entities
- Critical in information dissemination
- Causes the network to have small diameter

Examples
- www, internet..
- Social networks
- Collaboration networks
Log log plots

- On ipython notebook
Log log plots for power law are nice and straight.
Be careful with log log plots

• The “straight” part needs to extend quite a few orders of magnitude for the pattern to be significant
• Fitting the straight line to determine the right coefficient alpha is not trivial due to non-linear nature of data
• Beware: log-normal distributions can look similar to power law.
Mean degree in a power law distribution

• The mean is finite iff $\alpha > 2$
  – (On an infinite graph)

• On the www $\alpha$ is slightly larger than 2
Model of power law networks

• We want a model that can be used to create power law networks
• Preferably one that mimics creation of actual power law networks like www
  – Gives us some idea of how these networks were created
Preferential attachment mechanism

• Idea: older and established (popular) sites are likely to have more links to them (yahoo, google...)

• So how about: When a new page arrives, it links to older pages in proportion to their popularity

• When a new link is created on a new page, randomly to older pages with probability of hitting a page x proportional to current popularity of x (number of links to x)
Preferential attachment model

• Takes a parameter $p$ in $[0,1]$
• On a new page, create $k$ links as follows:
  • When creating a new link:
  • With probability $p$
    – Assign it with preferential attachment mechanism
  • With probability $1-p$
    – Assign it with uniform random probability to any existing page
Preferential attachment model

- Takes into consideration that popularity is not the only force behind link creation.
- The randomly assigned links model other reasons for link creation.
- Can be proven to produce power law. see [Kempe lecture notes, 2011]
- Produces same exponent as www for $p \sim 0.9$
- Let’s see in the data
Power law often appears in other places

• Popularity of books
• Popularity of people, songs, ....

• Preferential attachment & power law are often a signature of artificial selection and popularity
Other reasons for power law

• Optimization:
  – Power law found in linguistics (word lengths): most frequent words are short
    • Mandelbrot, Zipf: emerges from need for efficient communication

• Random processes:
  – Press space with probability $p$, else press a random letter key
  – This will produce a power law distribution of word lengths