**Notes 2 - Solutions**

PROBLEM 2:

Show that a connected graph has $\Omega(n)$ triads (counting both open and closed)

The above statement would be true if there was a constant $k > 0$ such that the number of triads $f(n)$ is always greater than $kf(n)$.

It is obviously true in the simple case of a graph with 3 nodes and a single open triad. In this case there is 1 triad and $k = \frac{1}{2}$ will work.

When we have a problem where we are trying to establish the number of something in general, and can see that it is true in a simple case, induction is often the way forward.

**Proof by Induction**

Base Case:

Let n=3 so that we have the minimum number of vertices required to form a graph with a triad. Let the graph have one open triad, so that it is minimally connected while still allowing triads.

Let $k = \frac{1}{2}$. Clearly, $f(3) = 1$. Therefore it is true in this case that:

$$f(3) \geq k * f(3)$$

as:

$$1 \geq \frac{1}{2}$$

Inductive Step:

We need to show that if the number of triads is $\Omega(n)$ for $n = j$, then it is also true for $n = j + 1$.

Say we have a graph with $j$ vertices and $f(j)$ triads such that $f(j) \geq \frac{1}{2}j$.

If we add a vertex then given that this is a connected graph there will necessarily be an edge connecting this vertex to the existing graph. Say the new vertex $x$ is connected to vertex $y$ which was included in the original $j$ vertexes. The number of triads can only increase as $y$ will be connected to at least one other vertex $z$ in the graph by the connectivity of the graph.

This implies that at least one new triad will be added.

Therefore $f(j+1) \geq f(j)+1$.

We also know that $f(j) \geq \frac{1}{2}j$ and clearly $\frac{1}{2}j+1 \geq \frac{1}{2}j+\frac{1}{2}$.

Combining these:

$$f(j+1) \geq \frac{1}{2}(j+1)$$

Note that if an edge was added instead of a vertex the number of triads could only increase, and so the same argument holds.

Therefore, for all $n \geq 3$ the number of triads is $\Omega(n)$ as there exists a value of $k$ such that $f(n) \geq kf(n)$.

> PROBLEM 3:
>
> What is the expected number of triangles or closed triads in the graph.

Say there is a probability $p$ of any two nodes being connected.

There are $\binom{n}{3}$ possible triangles. The probability of any given triangle is then $p^3$.

This implies that the expected number of triangles is $p^3\binom{n}{3}$

> PROBLEM 4:
>
> Clustering coefficient is the ratio of number of closed triads to number of all triads. Show that for ER graphs with $p = \frac{\ln(n)}{n}$ (where $n$ is an unknown variable) the clustering coefficient cannot be bounded from below buy a constant. (That is, there is no constant number such that CC is always greater than that.

As seen above, the expected number of closed triads is $p^3\binom{n}{3}$.

With $p = \frac{\ln(n)}{n}$ this gives an expected value of $\frac{\binom{n}{3}}{n^3}\ln^3(n)$.

As $\binom{n}{3}$ is $O(n^3)$, this implies that the expected number of closed triads is $O(\ln^3(n))$

In problem 1 we showed that a connected graph has $\Omega(n)$ triads.

Therefore the ratio is $\frac{O(\ln^3(n))}{\Omega(n)}$.

Regardless of which clustering coefficient we chose, we could always then choose a value of $n$ which was large enough to make $\frac{O(\ln^3(n))}{\Omega(n)}$ smaller than this constant as this ratio approaches 0 for large enough $n$. Thus, for these graphs, there is no constant that act as a lower bound for CC.