Small world networks

Social and Technological Networks

Rik Sarkar

University of Edinburgh, 2018.

Milgram’s experiment

- Take people from random locations in USA
- Ask them to deliver a letter to a random person in Massachusetts
- A person can only forward the letter to someone you know
- Question: How many hops do the letters take to get to destination?

Results

- Out of 296 letters, only 64 completed
- Number of hops varied between 2 and 10
- Mean number of hops 6
- There were a few people that were the last hop in most cases

Discussion of experiment

Discussion of experiment

- Short paths exist between pairs (small diameter)
- More surprisingly, people find these short paths
- Without knowing the entire network
- *Decentralized search*
- Analogous to routing without a routing table
- People use a “greedy” strategy
- Forward to the friend nearest to the destination

Recent results

- Milgrams results reproduced on better data
- Use online data (Livejournal, facebook)
- Containing approximate locations
- Simulate the process of forwarding letters
- Results similar to original experiment
- Relatively short diameter, successful decentralized search

In popular culture

- Erdos distance
- Kevin bacon distance

Definition of small worlds

- Small diameter
- Large clustering coefficient
  - Related to homophily — similar people connect to each-other
  - “Similar”: close in some coordinate value (or metric)
- Supports decentralized search
  - People find short paths without knowing the entire network
- (Usually) High expansion

Model 1: Watts and Strogatz

Nature 1998

- Parameters n, k, p $n > k > \ln n$
  - Often k is taken to be a constant in practice with the idea that people cannot have infinitely large friend-circles
- Put nodes in a ring of size n
- Connect each to k/2 neighbors on each side

Model 1: Watts and Strogatz

Nature 1998

- Parameters n, k, p $n > k > \ln n$
  - Often k is taken to be a constant in practice with the idea that people cannot have infinitely large friend-circles
- Put nodes in a ring of size n
- Connect each to k/2 neighbors on each side

- What is the diameter and CC?

Model 1: Watts and Strogatz

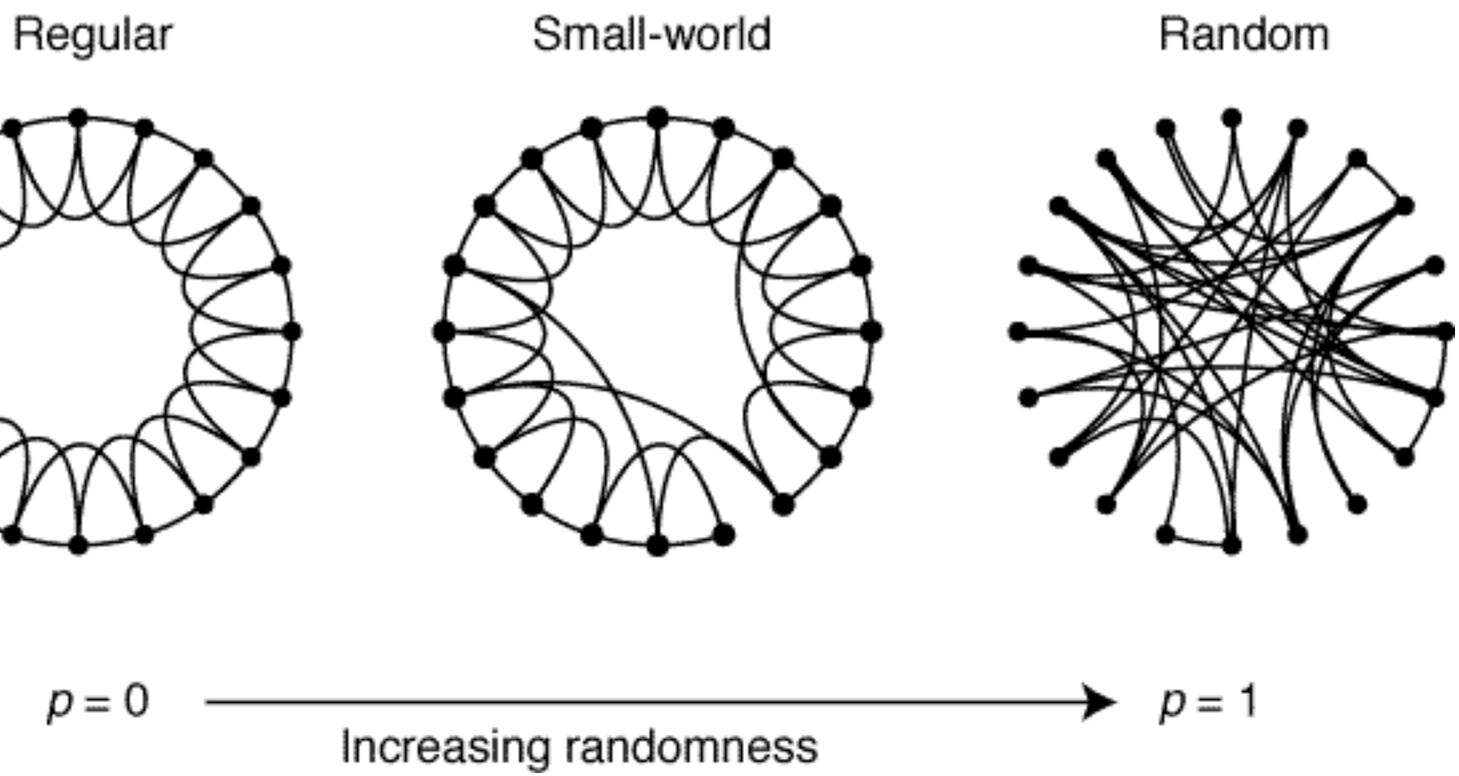
Nature 1998

- Parameters n, k, p $n > k > \ln n$
  - Often k is taken to be a constant in practice with the idea that people cannot have infinitely large friend-circles
- Put nodes in a ring of size n
- Connect each to k/2 neighbors on each side

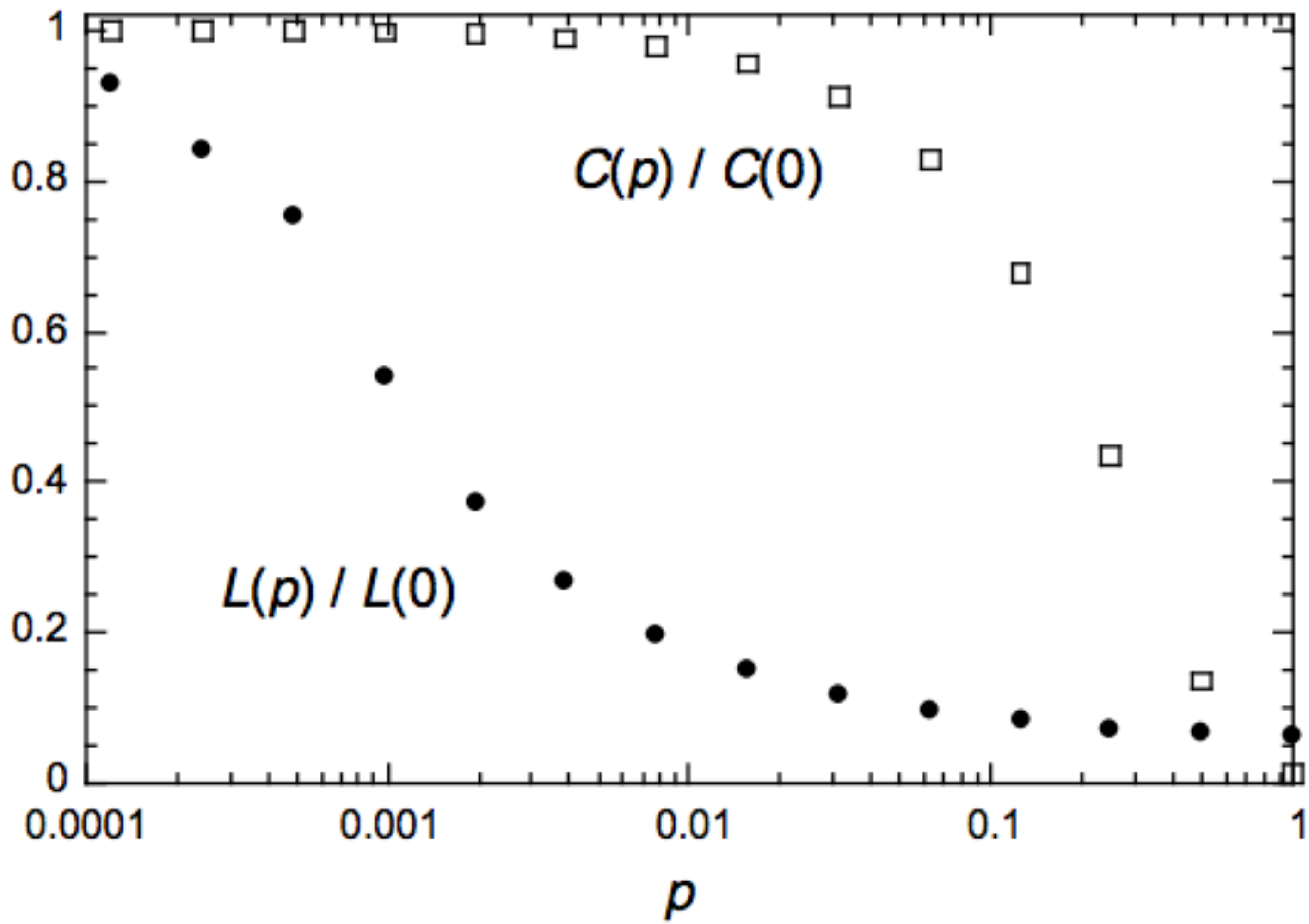
- With probability p rewire each edge of a vertex to a random vertex

Small world

- In between random and structured



Small world



Properties

- Average clustering coefficient per vertex bounded away from zero
  - In other words: at least a constant
- Connected: sufficient random edges + regular edges
- Short diameter

Watts-strogatz model does not explain milgram’s experiment

Watts-strogatz model does not explain milgram's experiment

- Milgram's experiment was on 2D plane
- Watts strogatz does not support decentralized search

Watts-strogatz model does not explain milgram's experiment

- Milgram's experiment was on 2D plane
- Watts-strogatz does not support decentralized search (poly(log n)) steps to destination

Outline

- A good small world model should support (poly(log n)) steps to destination
- We will show that watts strogatz needs poly(n)

- Kleinberg's model supports $(\log(n))^2$ step route
  - Show that the route can be divided into O(log n) phases
  - Each phase takes O(log n)

Decentralized search in random link networks

- Decentralized search does not work to produce short paths
- Let us consider 2D (n x n grid):
  - We want to show that if every node works only on its local information (edges it has)
- Then there is no algorithm that delivers the message in less than poly(n) messages.

Decentralized search in random link networks

- Consider s and t separated by $\Omega(n)$ hops
- Take ball B of extrinsic radius around $n^{2/3}$ t
  - There are $O(n^{2/3})^2$ nodes in B
- When we are already at distance $n^{2/3}$ (on the edge of B)
  - A long link can help only if it falls inside B
- Otherwise we take a step along a short link
- What is the probability that a random link from s hits B?
- This is ~ $O((n^{2/3})^2/n^2) = O(n^{-2/3})$
- The expected number of steps before getting a useful long link is : $\Omega(n^{2/3})$

Decentralized search

- Therefore long links are not really useful in reaching t
- The number of steps is poly(n).

Model 2 : Kleinberg's model

STOC 2000, Nature 2000, ICM 2006

- Idea: Long links are not helping much
  - Getting closer to the destination does not increase the chances of getting a long link close to destination.
- Make the probability of a long link sensitive to the distance
  - Nearby nodes are more likely to have a long link

Model 2 : Kleinberg’s model

- Suppose $d(u, v)$ is the extrinsic distance between nodes u and v in the plane
- Then u connects its long link to v
- with probability $\propto \frac{1}{d(u,v)^{\alpha}}$

Kleinberg’s model

- Links to nearby nodes are more likely
  - A node knows more people locally
  - With increasing distance, it knows fewer and fewer people
  - At the largest scale it knows only a handful
  - More representative of how people have their contacts spread
- We want to show that the model permits short paths to be found

The proportionality constant

$$\Pr[(u,v)] = \frac{1}{\gamma}\frac{1}{d(u,v)^{\alpha}}$$

$$\alpha = 2 \Rightarrow \gamma = \Theta(\ln n)$$

- Sketch of proof: Take rings of thickness 1 at distances 1,2,3…
- The number of nodes at distance d ~ Θ(d)
- Thus from any node:

$$\frac{1}{\gamma}\sum_{d=1}^{n} d^{-2}\Theta(d) = 1$$

$$\frac{1}{\gamma}\Theta\left(\sum_{d=1}^{n}\frac{1}{d}\right) = 1$$

$$\Rightarrow \frac{1}{\gamma}\Theta(\ln n) = 1$$

Theorem

$$\alpha = 2$$

- Permits finding $O(\log^2 n)$ intrinsic length paths
- Using local routing : Always move to the neighbor nearest to the destination

Proof

- Main idea:
- In O(log n) steps, the extrinsic distance is halved
  – Let us call this one phase
- In O(log n) phases, the distance will be 1
- So, we need to show the first claim: one phase lasts O(log n) steps

One Phase lasts log n steps

- Suppose distance from s to t is d
- take ball B of radius d/2 around t
- There are about $\Theta(d^2)$ nodes in this area
- The probability that a long link hits B is

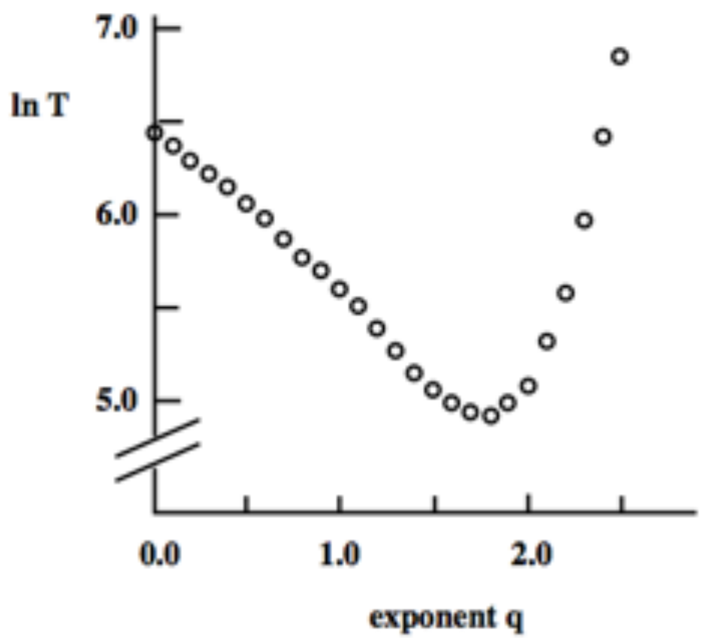
$$\frac{1}{\Theta(\log n)} \sum_{v \in B} d(s,v)^{-2} \geq \Theta\left(\frac{1}{\log n} d^2 d^{-2}\right) = \Theta\left(\frac{1}{\log n}\right)$$

One phase lasts log n steps

- Thus, the expected number of steps before we find a link into B is log n.
- And there are log n such phases
- Therefore, this method finds a path of $\log^2 n$ steps

Other exponents

- < 2 : more like uniform random
- > 2 : Shorter links, almost same as basic grid..



Generality

- Search is a very general problem
- Search for an item, search for a path, search for a set, search for a configuration
- Decentralized: Operation under small amount of information. (local, easy to distribute)

Small worlds in other networks

- Brain neuron networks
- Telephone call graphs
- Voter network
- Social influence networks …

- Applications:
- Peer to peer networks
- Mechanisms for fast spread of information in social networks
- Routing table construction