Growth and Expansion

Social and Technological Networks

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Project

- Suggested list to be announced Thursday/Friday (11^{th}/12^{th})
  - On Email and Piazza
- Select project by Thursday (18^{th})
  - You can work in groups of 1,2 or 3
  - But Must submit individual reports
  - The group is for discussion and possibly some common tasks. But marking is individual.
- A short (half page) proposal due around October 26.
  - Not graded. Feedback only.
- Final report due around Nov 15.

Network construction

- Given any dataset with distances between items, we can construct a network

Computing distances for categorical data

- Suppose we are given categorical data
- E.g.
  - We are given list of clubs people belong to
  - Or list of songs they like etc..
- Cosine distance
  - Represent the list as 0-1 vectors A, B, …
  - Find cosine similarity $S_c = \dfrac{\mathbf{A} \cdot \mathbf{B}}{\|\mathbf{A}\|_2 \|\mathbf{B}\|_2}$
  - Cosine distance $d_c = 1 - S_c$

Computing distances for categorical data

- Jaccard similarity:
  - Treat the vectors as sets A, B..
  - And compute $$J(A,B) = \frac{|A \cap B|}{|A \cup B|}$$
  - Distance J_d = 1-J

Computing distances for categorical data

- Min category distance
  - Take the size of the smallest club with both A and B as the distance

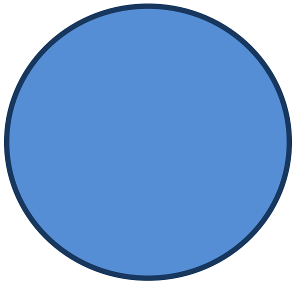
- You can come up with many other ways of computing distance

Finding distance between two nodes in a graph

- Breadth first search
- Dijkstra’s shortest path algorithm

Ball

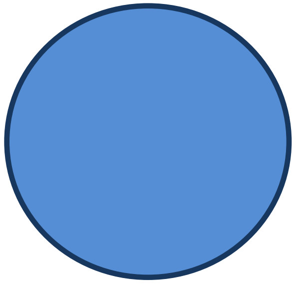
- A ball of radius r at point v:
  - The set of all points within distance r from v
  - Called a disk in 2D
- Usually written as
  - B(v,r) or
  - $B_r(v)$



- Sphere $S_r(v)$: set of points at distance exactly r from v
  - The boundary of the ball
  - 1-D sphere: boundary of a 2-D ball
  - 2-D sphere: boundary of a 3-D ball etc

Size of a ball

- The “measure” in a suitable dimension
  - Area in 2D
  - Volume in 3D etc
  - What about 1D?



- What is the measure of a sphere?
  - 1D?
  - 2D?

Growth of a metric

- How does the size of a ball B(v,r) grow with radius?
- In 1D?
- In 2D?
- In 3D?

Growth of Euclidean metric

- D-dim

$$\Theta(r^d)$$

Growth can be used to detect dimension

- E.g. Long strips

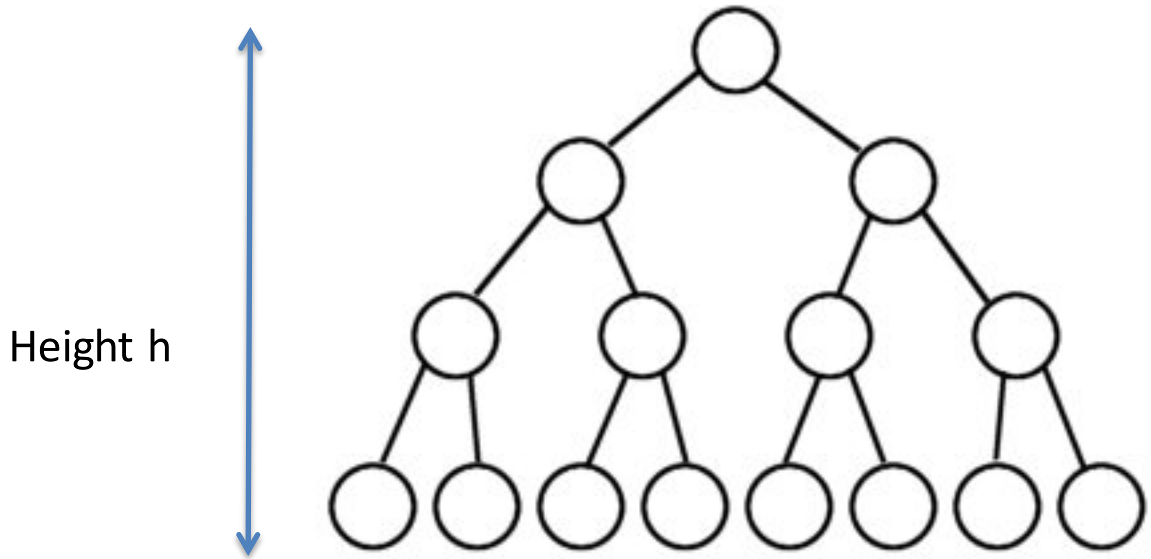
  - Growth is linear: O(r)
  - Compared to size, this is 1-D

Ball B(v,r) in a graph

- The set of nodes within distance r of v
  - Sphere: Nodes at distance exactly r
- Measure or volume:
  - The number of nodes in $B_G(v, r)$ [Ball in metric G]
  - The number of nodes in $B_E(v, r)$ [Ball in Euclidean metric]

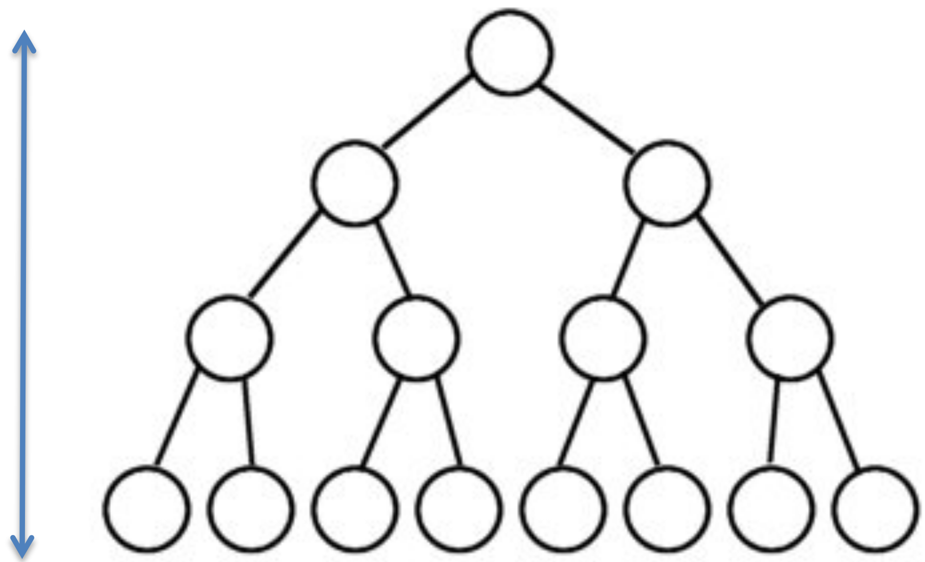
- Growth
  - How does $|B_G(v, r)|$ grow with r
  - For Chain?
  - Cycle?
  - Grid?
  - Balanced binary tree?
- How do these change for $B_E(v, r)$?

Growth for balanced binary trees

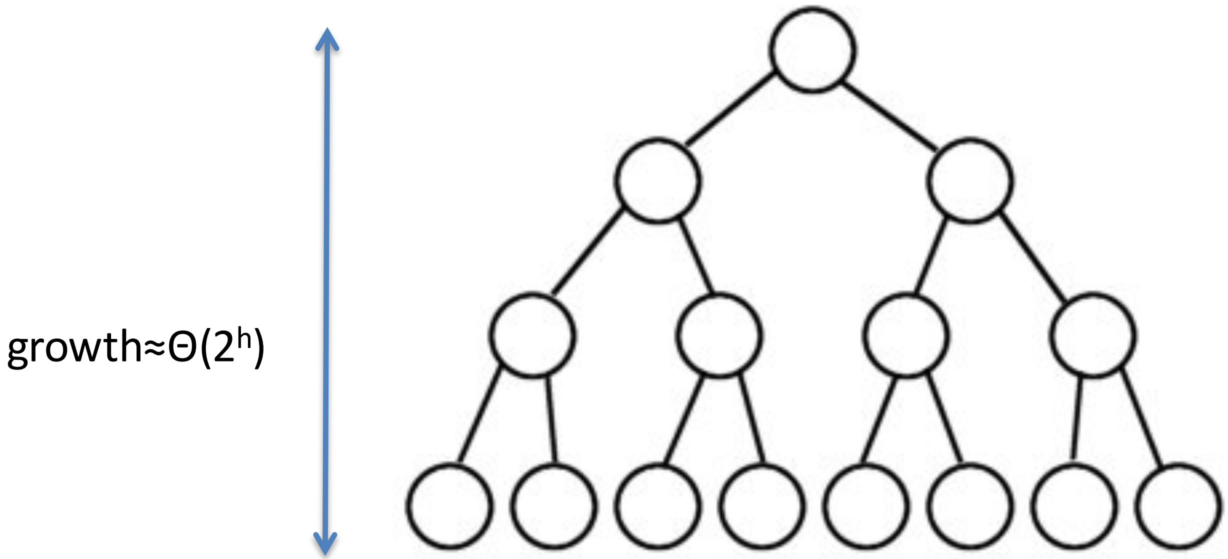


Growth for balanced binary trees

$n=2^h$



Growth for balanced binary trees



What are the diameters of these graphs

- Finite chain
- Finite Grid
- Balanced binary tree

- Intrinsic metric
- Extrinsic metric

How does the sphere grow?

- For chain
- Grid
- Balanced binary tree

- Graph metric
- Euclidean metric

Doubling dimension

- A metric space X is said to have constant doubling dimension if
  - Any ball $B_X(v, r)$ can be covered by
  - At most M balls of radius r/2
  - For some constant M
- lg M is called the doubling dimension of X
  - (lg is log base 2)

- Measure of dimension in arbitrary structures, like graphs

Edge Expansion of Graphs

- How fast the 'boundary' expands relative to 'volume' or 'size' of a subset
- Boundary of node set S :
  - $e^{out}(S)$: edges with exactly one end-point in S
- Expansion:

$$\alpha = \min_{S \subseteq V} \frac{|e^{out}(S)|}{\min(|S|, |\overline{S}|)}$$

Expansion

$$\alpha = \min_{S \subseteq V} \frac{|e^{out}(S)|}{\min(|S|, |\overline{S}|)}$$

- Equivalently:

$$\alpha = \min_{|S| \leq n/2} \frac{|e^{out}(S)|}{|S|}$$

Expanders

- A class of graphs with expansion at least a constant

$$\alpha \geq c$$

  - For some constant c

Examples of expanders

- Random d-regular graphs for d>3
  - All nodes have degree d
  - Nodes are connected at random

Configuration model of Random graphs

- Suppose we want a graph that is random
- But has given degree for each vertex:

$$d_1, d_2, d_3, \dots d_n$$

- At each vertex i we d_i *open-edges*
- Pair up the edges randomly
- If all degrees = d
  - Graph is called d-regular

Expanders have small diameter

- A graph with degrees ≤ d and expansion ≥ α
- Has diameter

$$O(\frac{d}{\alpha}\lg n)$$

Other properties

- Expanders are well connected
- Usually sparse (number of edges much smaller than n^2)
- Diffusion processes spread fast in an expander
- Random walks mix fast (achieve steady state)
- …