Social and Technological Networks Edinburgh, 2018

Exercises 1. Random graphs - solutions.

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Q 1. *Show that a connected graph has $\Omega(n)$ triads (counting both open and closed).*

Answer. As per the definition of $\Omega(n)$, we need to show that for some constant $c > 0$, number of triad in a connected graph $T > c.n$ for $n > n_0$.

We would prove this by induction. In a minimally connected graph, a tree with three vertices, the number of triad is one. This constitutes the base case with $c = 1/3$.

For induction, we are assuming there is a graph G with n nodes and T triads. In G, $T \geq \frac{n}{3}$. Now, we'll show that after adding nodes and edges to G, it still holds this property.

- Adding an edge to G: Adding an edge to G only increases the number of triads. Thus, the number of triads in the new graph $T' \geq T \geq \frac{n}{3}$.
- Adding an vertex to G: As the new graph is to be connected, there should be at least an edge connecting the newly added vertex (i) to one of the vertices (j) in G. As, G was connected there was at least an edge jk. Thus, ijk is a triad in the new graph. Thus, $T' \geq T + 1 \geq \frac{n}{3} + 1 > \frac{1}{3}(n + 1)$.

Therefore, for $n \geq 3$, $T \geq c.n$ where $c = \frac{1}{3}$.

In class we saw that a triangle or closed triad is is three vertices a, b, c with all edges ab, bc, ca between them. The number of possible triangles or triads is clearly $\binom{n}{3}$, which is $\Theta(n^3)$. The probability that a particular triangle exists is p^3.

Q 2. *What is the expected number of triangles or closed triads in the graph?*

Answer. The number of possible triangles is $\binom{n}{3}$. Each exists with probabilty p^3. Thus the expected number of triangles is $\binom{n}{3}p^3$.

Q 3. *Clustering coefficient is the ratio of number of closed triads to number of all triads. Show that for ER graphs with $p = \frac{\ln n}{n}$ (where n is an unknown variable) the clustering coefficient cannot be bounded from below buy a constant. (That is, there is no constant number such that CC is always greater than that.)*

Answer. The expected number of triangles is $\binom{n}{3}p^3 = O(\ln^3 n)$. Thus the expected clustering coefficient is $\frac{O(\ln^3 n)}{\Omega(n)}$. For any constant c, it is possible to select a large enough n such that this ratio is smaller than c.

In class slides, we had an expression that at limit $n \to \infty$, clustering coefficient is zero for ER graph at $p = (\ln n)/n$, which also follows from the same result.

(Check that these statements are true, given the definitions of O and Ω).

Q 4. *Write code in your iPython notebook to create the threshold plot for isolated vertices or giant components. How does the look of the plot change with number of vertices in the graph?*