Social and Technological Networks Edinburgh, 2017

Exercises 1 solutions. Sample problems.

Rik Sarkar *Exercises*

The following are sample problems to test your background for the course. You should be able to solve most of these without looking up references (or, without looking up too many of them). At least solve the easy ones, and you should have some idea of what the harder ones mean and how to approach them.

Exercise 0.1. *How many edges can a graph have? (assuming there is at most one edge between any two vertices.) If each possible edge exists with a probability* p*, what should be the value of* p *such that the expected number of edges at each vertex is* 1*?*

Answer. Assuming it is a simple graph, there is at most one edge between any pair of nodes. And there are $\binom{n}{2}$ nodes. Thus a grpah can have $\binom{n}{2} = \frac{n(n-1)}{2}$ edges.

A node v can have at most $n-1$ edges incident on it. Each of these exists with a probability p independent of the others. The expected number of edges at node v is $(n-1)p$. Therefore we can solve for p from $(n-1)p = 1$, therefore $p = \frac{1}{n-1}$.

Exercise 0.2. *Suppose every year Mr. X makes double the number of friends he made last year (starting with making 1 friend in first year). In how many years will he make* n *friends? (asymptotic notation is fine.)*

Answer. Mr. X makes 1 friend in the first year, 2 in the second year, so he has in total $1+2$ friends in the second year. At the end of m-th year he will have $1+2+\ldots 2^{m-1} = 2^m - 1$ friends. Now let us select the smallest m such that $2^m - 1 \geq n$. Observe that by this definition, after year $m-1$, he had strictly less than n friends, and after year m he can actually have much more than n friends. However, m is still the right answer, because we are counting whole years.

Expressing m in terms of n, we have $m = \lceil \lg(n+1) \rceil$ [1]. We have to use the ceiling function here because $n+1$ may not be a power of 2, and we need to take the next integer to get a proper count.

Exercise 0.3. *Suppose we throw* k *balls into* n *bins randomly, what is the probability that bin* 1 *remains empty?*

Answer. $\Pr[\text{bin 1 is empty after 1 throw}] = 1 - \frac{1}{n}$. Therefore, $\Pr[\text{bin 1 is empty after k throw}] = (1 - \frac{1}{n})^k$.

[1] The $\lceil \bullet \rceil$ symbol stands for the function *ceiling* implying the integer greater than or equal to its argument.

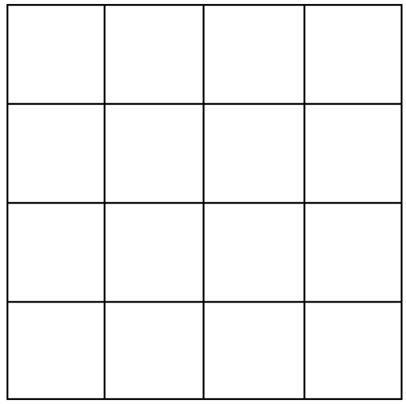

Figure 1. Example of a Grid.

Exercise 0.4. *A grid is an arrangement of squares as shown in Fig.* **??**. *Prove that for any given grid, the number of grid squares inside a circle of radius r is $O(r^2)$.*

Proof: Let us suppose each grid square has side s, and area s^2. Since the interiors of the grid squares are disjoint, the total area covered by any n distinct grid squares is ns^2. The area of the circle of radius r is πr^2, and the maximum number of possible squares in the circle is $\leq \pi r^2/s^2$. For a given grid s is fixed, so the number of squares in the circle is $O(r^2)$. $\square$

Exercise 0.5. *Show that a bipartite graph has no cycles of odd length.*

Proof: Suppose the two partitions are U and V. Without loss of generality, let us suppose that the cycle C starts from $u \in U$. By definition of a bipartite graph, traversal along C must alternate between the $U \to V$ type on odd numbered edges and the $V \to U$ type on even numbered edges. Since the cycle must end at $u \in U$, it must end with a $V \to U$ type edge which is even numbered. Thus C must have even numbered edges. $\square$

Exercise 0.6. *An* isolated *vertex is one which has no edges. Consider a graph G with n vertices such that every edge exists with probability $p = (1+\varepsilon)(\ln n)/(n-1)$. Prove that the probability that G has one or more isolated vertices is less than $1/n^{\varepsilon}$.*

[Hint: Write the probability that none of the possible edges at a vertex exist. Use the inequality $(1-p)^{1/p} \leq 1/e$ for $0 \leq p \leq 1$. You can also use the Union bound, which says $\Pr A \text{ OR } B \leq \Pr A + \Pr B$.]

Proof: At vertex v, probability that a particular edge does not exist is $(1-p)$; the probability q that the vertex v is isolated, i.e. all $n-1$ possible edges do not exist is $q = (1-p)^{n-1}$. We can substitute $n-1 = (1+\varepsilon)(\ln n)/p$ in the exponent, and get $q \leq e^{-((1+\varepsilon)(\ln n))}$. Therefore, $q \leq n^{-(1+\varepsilon)}$. $\square$

Harder problems:

*** Exercise 0.7.** *Show that the matrix* $M = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$ *has orthogonal eigenvectors for any real numbers* a, b. *[Hint: Try comparing values of* $(Mv) \cdot u$ *and* $(Mu) \cdot v$ *for vectors* u *and* v*, then use definition of eigen vectors. You can use the fact that* M *has eigen values* λ *and* μ *that are distinct.]*

Proof: $(Mv) \cdot u = \begin{pmatrix} av_1 + bv_2 \\ bv_1 + av_2 \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = av_1u_1 + bv_2u_1 + bv_1u_2 + av_2u_2.$

And $(Mu) \cdot v = \begin{pmatrix} au_1 + bu_2 \\ bu_1 + au_2 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = au_1v_1 + bu_2v_1 + bu_1v_2 + au_2v_2.$

Thus $(Mv) \cdot u = (Mu) \cdot v$ for any vectors u and v. Now suppose u and v are eigen vectors of M, with eigen values λ and μ. Then $(\lambda u) \cdot v = (Mu) \cdot v = (Mv) \cdot u = (\mu v) \cdot u$.

Since $\lambda \neq \mu$, it follows that $(u \cdot v)(\lambda - \mu) = 0$ implies $u \cdot v = 0$, that is, u and v are orthogonal. $\square$

*** Exercise 0.8.** *Let us define matrices* A *and* B *to be* similar *if there exists a matrix* P *such that* $A = PBP^{-1}$.

For similar matrices A *and* B*, show that if* λ *is an eigenvalue of* A*, then it is also an eigenvalue of* B*. [Hint: Use definition of eigen vector, then multiply both sides by suitable matrices. The eigen vectors corresponding to the eigen value may not be the same. You can assume* A, B, P *are square.]*

Proof: Let x be an eigen vector of A with eigen value λ. Also, let use denote $P^{-1}x = y$.

$$\begin{aligned} Ax &= \lambda x \\ \Rightarrow PBP^{-1}x &= \lambda x \\ \Rightarrow BP^{-1}x &= P^{-1}\lambda x = \lambda P^{-1}x \text{ [After Left-multiplication by } P^{-1}\text{]} \\ \Rightarrow By &= \lambda y \text{ [Substituting } y.\text{]} \end{aligned}$$

Therefore, λ is also an eigen value of B with eigen vector y. $\square$