Social and Technological Networks Edinburgh, 2016

Exercises 1. Sample problems.

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The following are sample problems to test your background for the course. You should be able to solve most of these without looking up references (or, without looking up too many of them). At least solve the easy ones, and you should have some idea of what the harder ones mean and how to approach them.

Exercise 0.1. *How many edges can a graph have? (assuming there is at most one edge between any two vertices.) If each possible edge exists with a probability p, what should be the value of p such that the expected number of edges at each vertex is 1?*

Exercise 0.2. *Suppose every year Mr. X makes double the number of friends he made last year (starting with making 1 friend in first year). In how many years will he make n friends? (asymptotic notation is fine.)*

Exercise 0.3. *Suppose we throw k balls into n bins randomly, what is the probability that bin 1 remains empty?*

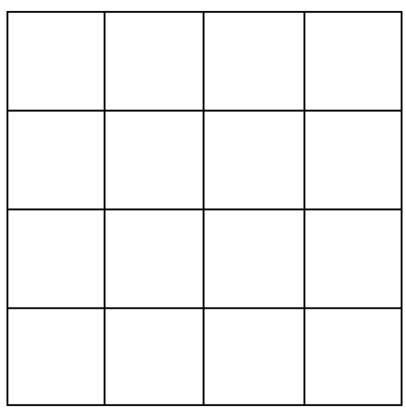

Figure 1. Example of a Grid.

Exercise 0.4. *A grid is an arrangement of squares as shown in Fig. 1. Prove that for any given grid, the number of grid squares inside a circle of radius r is $O(r^2)$.*

Exercise 0.5. *Show that a bipartite graph has no cycles of odd length.*

Exercise 0.6. *An* isolated *vertex is one which has no edges. Consider a graph G with n vertices such that every edge exists with probability $p = (1+\varepsilon)(\ln n)/(n-1)$. Prove that the probability that G has one or more isolated vertices is less than $1/n^{\varepsilon}$.*

[Hint: Write the probability that none of the possible edges at a vertex exist. Use the inequality $(1-p)^{1/p} \leq 1/e$ for $0 \leq p \leq 1$. You can also use the Union bound, which says $\Pr A \text{ OR } B \leq \Pr A + \Pr B$.]

Harder problems:

* **Exercise 0.7.** *Show that the matrix $M = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$ has orthogonal eigenvectors for any real numbers a, b. [Hint: Try comparing values of $(Mv) \cdot u$ and $(Mu) \cdot v$ for vectors u and v, then use definition of eigen vectors. You can use the fact that M has eigen values λ and μ that are distinct.]*

* **Exercise 0.8.** *Let us define matrices A and B to be* similar *if there exists a matrix P such that $A = PBP^{-1}$.*

For similar matrices A and B, show that if λ is an eigenvalue of A, then it is also an eigenvalue of B. [Hint: Use definition of eigen vector, then multiply both sides by suitable matrices. The eigen vectors corresponding to the eigen value may not be the same. You can assume A, B, P are square.]