Spectral analysis of ranking algorithms

Rik Sarkar

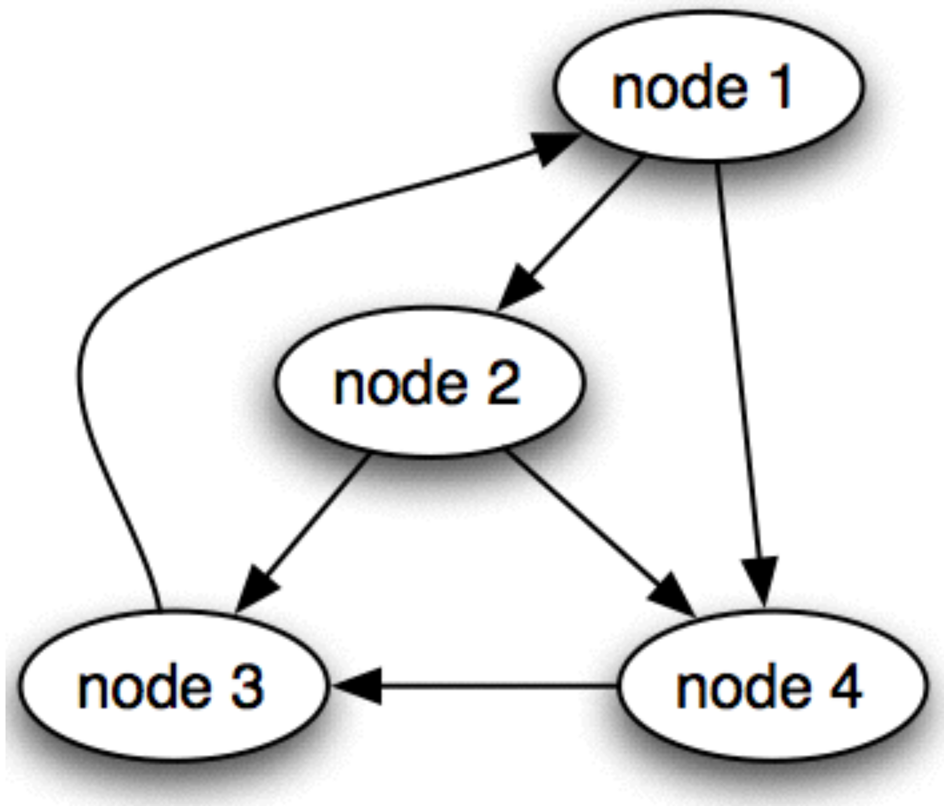
- **No Class on Friday 23rd October**

- Projects will be announced later today

Recap: HITS algorithm

- Evaluate hub and authority scores
- Apply Authority update to all nodes:
  - auth(p) = sum of all hub(q) where q -> p is a link
- Apply Hub update to all nodes:
  - hub(p) = sum of all auth(r) where p->r is a link
- Repeat for k rounds

Adjacency matrix



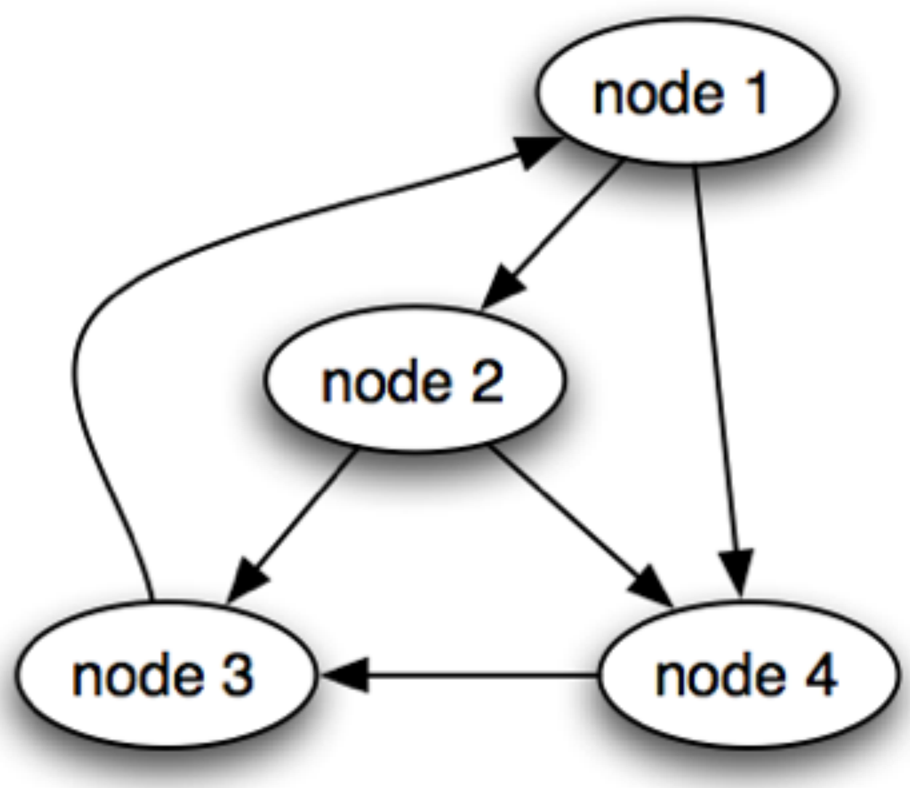
$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Hubs and authority scores

- Can be written as vectors h and a
- The dimension (number of elements) of the vectors are n

Update rules

- Are matrix multiplications:
- $$h \leftarrow Ma$$



$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \\ 2 \\ 4 \end{bmatrix}$$

- Hub rule for i : sum of a-values of n*odes that i points to:*

$$h \leftarrow Ma$$

- Authority rule for i : sum of h-values of *nodes that point to i:*

$$a \leftarrow M^T h$$

Iterations

- After one round:

$$a^{\langle 1 \rangle} = M^T h^{\langle 0 \rangle}$$

$$h^{\langle 1 \rangle} = M a^{\langle 1 \rangle} = M M^T h^{\langle 0 \rangle}$$

- Over k rounds:

$$h^{\langle k \rangle} = (M M^T)^k h^{\langle 0 \rangle}$$

Convergence

- Remember that h keeps increasing
- We want to show that the normalized value

$$\frac{h^{\langle k\rangle}}{c^k} = \frac{(MM^T)^k h^{\langle 0\rangle}}{c^k}$$

- Converges to a vector of finite real numbers as k goes to infinity
- If convergence happens: $$(MM^T)h^{\langle * \rangle} = ch^{\langle * \rangle}$$

Eigen values and vectors

$$(MM^T)h^{\langle * \rangle} = ch^{\langle * \rangle}$$

- Implies that for matrix (MM^T)
  - *c* is an eigen value, with
  - $h^{\langle * \rangle}$ as the corresponding eigen vector

Proof of convergence to eigen vectors

- Theorem: A symmetric matrix has orthogonal eigen vectors. (see sample problems from lecture 1)
  - They form a basis of n-D space
  - Any vector can be written as a linear combination
- (MM^T) is symmetric

- Suppose sorted eigen values are:

$$|c_1| \geq |c_2| \geq \cdots \geq |c_n|$$

- Corresponding eigen vectors are:

$$z_1, z_2, \ldots, z_n,$$

- We can write any vector x as

$$x = p_1 z_1 + p_2 z_2 + \cdots + p_n z_n$$

- So:

$$\begin{aligned}(MM^T)x &= (MM^T)(p_1 z_1 + p_2 z_2 + \cdots + p_n z_n) \\ &= p_1 MM^T z_1 + p_2 MM^T z_2 + \cdots + p_n MM^T z_n \\ &= p_1 c_1 z_1 + p_2 c_2 z_2 + \cdots + p_n c_n z_n,\end{aligned}$$

$$
\begin{aligned}
(MM^T)x &= (MM^T)(p_1z_1 + p_2z_2 + \cdots + p_nz_n) \\
&= p_1MM^Tz_1 + p_2MM^Tz_2 + \cdots + p_nMM^Tz_n \\
&= p_1c_1z_1 + p_2c_2z_2 + \cdots + p_nc_nz_n,
\end{aligned}
$$

- Over k iterations:

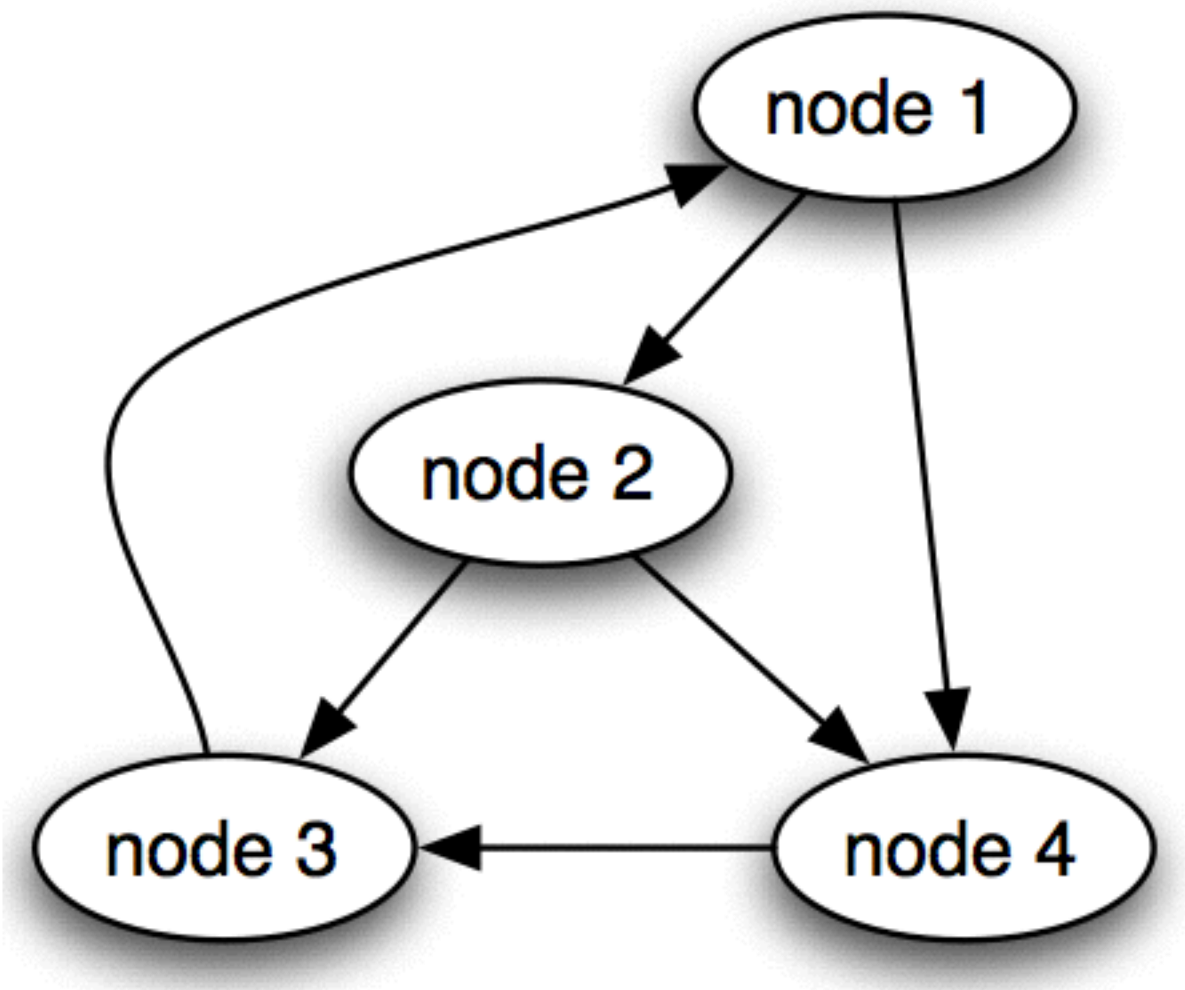
$$
(MM^T)^kx = c_1^kp_1z_1 + c_2^kp_2z_2 + \cdots + c_n^kp_nz_n
$$

- For hubs: $h^{\langle k\rangle} = (MM^T)^kh^{\langle 0\rangle} = c_1^kq_1z_1 + c_2^kq_2z_2 + \cdots + c_n^kq_nz_n$
  - So: $\dfrac{h^{\langle k\rangle}}{c_1^k} = q_1z_1 + \left(\dfrac{c_2}{c_1}\right)^kq_2z_2 + \cdots + \left(\dfrac{c_n}{c_1}\right)^kq_nz_n$
- If $|c_1| > |c_2|$, only the first term remains.
- So, $\dfrac{h^{\langle k\rangle}}{c_1^k}$ converges to q_1z_1

Properties

- The vector $q_1 z_1$ is a simple multiple of z_1
  - A vector essentially similar to the first eigen vector
  - Therefore independent of starting values of h
- q1 can be shown to be non-zero always, so the scores are not zero
- Authority score analysis is analogous

Pagerank Update rule as a matrix derived from adjacency



$$
\begin{bmatrix}
0 & 1/2 & 0 & 1/2 \\
0 & 0 & 1/2 & 1/2 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
$$

$$r \leftarrow N^T r$$

- Scaled pagerank:

$$r \leftarrow \tilde{N}^T r$$

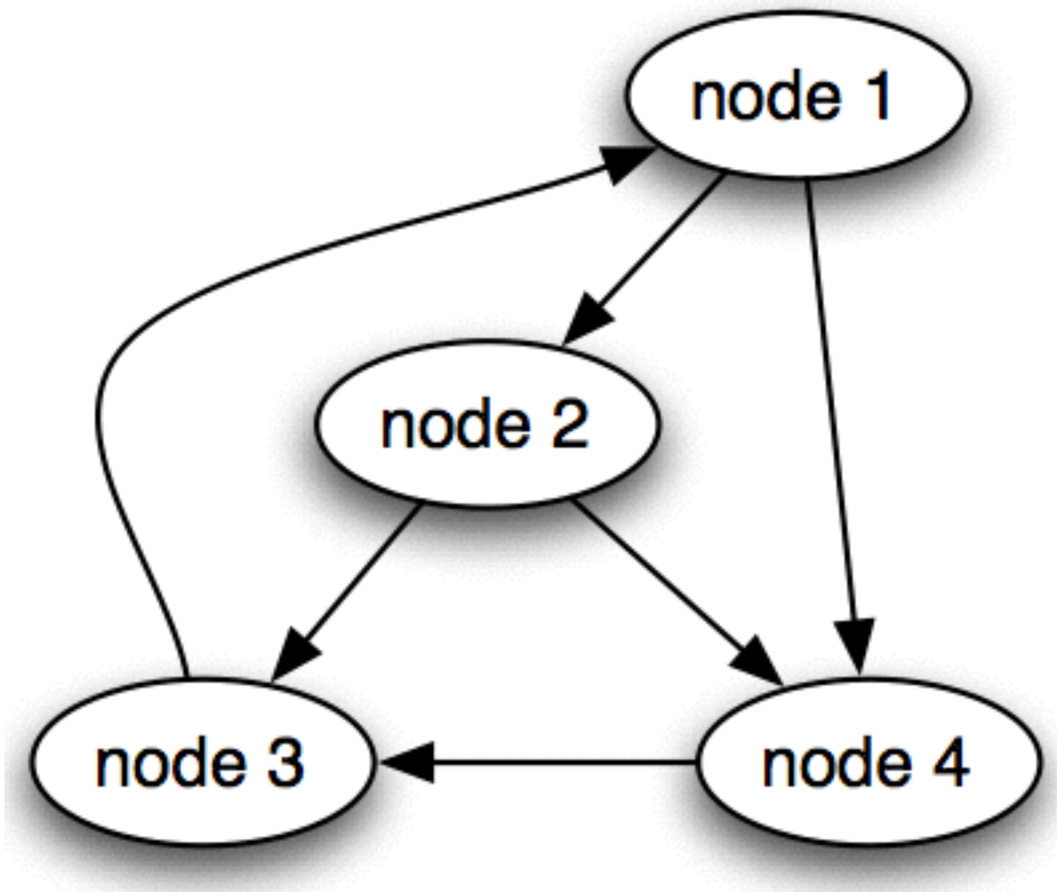
- Over k iterations:

$$r^{\langle k \rangle} = (\tilde{N}^T)^k r^{\langle 0 \rangle}$$

- Pagerank does not need normalization.

$$\tilde{N}^T r^{\langle * \rangle} = r^{\langle * \rangle}$$

- We are looking for an eigen vector with eigen value=1



$$\begin{bmatrix} .05 & .45 & .05 & .45 \\ .05 & .05 & .45 & .45 \\ .85 & .05 & .05 & .05 \\ .05 & .05 & .85 & .05 \end{bmatrix}$$

- For matrix P with all positive values, Perron's theorem says:
    - A unique positive real valued largest eigen value c
    - Corresponding eigen vector y is unique and has positive real coordinates
    - If c=1, then $P^k x$ converges to y

Random walks

- A random walker is moving along random directed edges
- Suppose vector b shows the probabilities of walker currently being at different nodes
- Then vector $N^T b$ gives the probabilities for the next step

Random walks

- Thus, pagerank values of nodes after k iterations is equivalent to:
  - The probabilities of the walker being at the nodes after k steps
- The final values given by the eigen vector are the steady state probabilities
  - Note that these depend only on the network and are independent of the starting points

History of web search

- YAHOO: A directory (hierarchic list) of websites
  - Jerry Yang, David Filo, Stanford 1995
- 1998: Authoritative sources in hyperlinked environment (HITS), symposium on discrete algorithms
  - Jon Kleinberg, Cornell
- 1998: Pagerank citation ranking: Bringing order to the web
  - Larry Page, Sergey Brin, Rajeev Motwani, Terry Winograd, Stanford techreport

Spectral graph theory

- Undirected graphs
- Diffusion operator
  - Describes diffusion of stuff — step by step
  - Stuff at a vertex uniformly distributed to neighbors — in every step

Laplacian matrix

- $L = D - A$
  - A is adjacency matrix
  - D is diagonal matrix of degrees

Example

Properties

- L is symmetric
- L is positive semidefinite (all eigen values are >= 0)
- Smallest eigen value $\lambda_0 = 0$
- Smallest non-zero eigen value: spectral gap $\lambda_1 - \lambda_0$
  - Determines the speed of convergence of random walks and diffusions
- Number of zero eigen values is number of connected components