# Diffusion and Epidemics

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# Spread of diseases

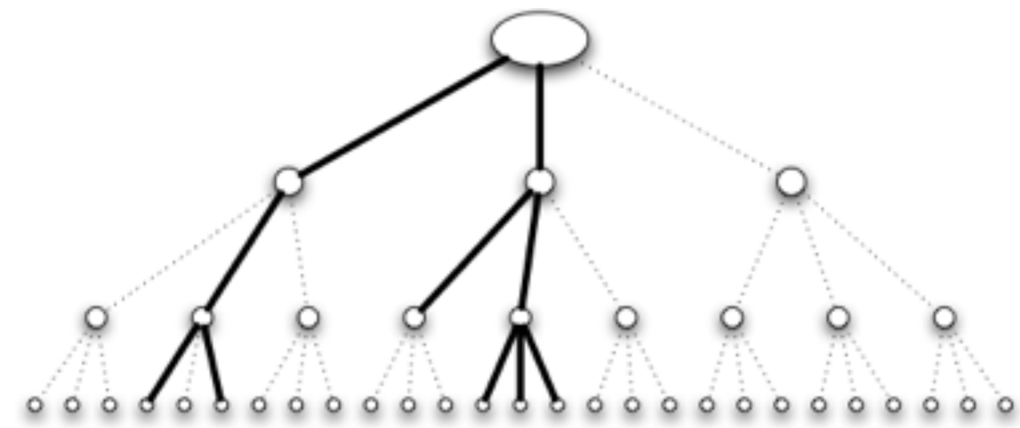
- Pattern depends on network structure
- e.g. spread of flu
- Network of people
- Network of airlines

- Different from idea/innovation contagion
  - Does not need a “decision”
  - Does not need multiple support
    - Infectious disease passes easily with some probability

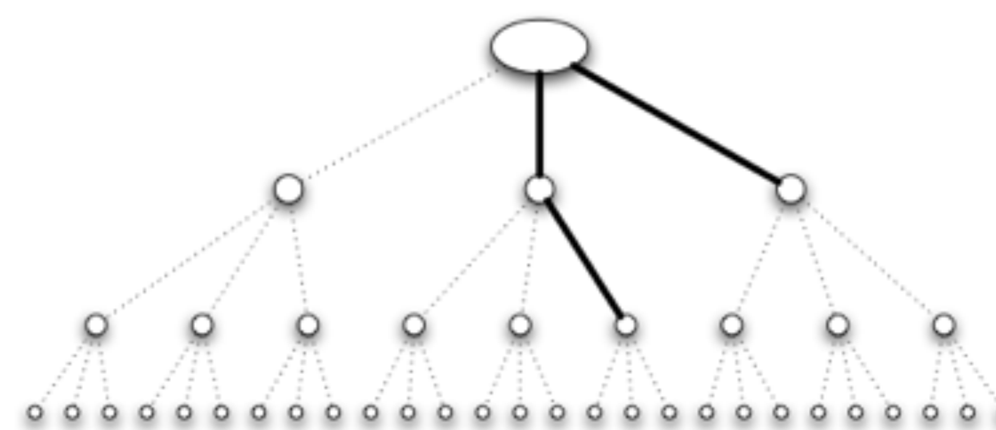
- Suppose everyone meets k new people and infects each with probability p

- That is, they infect R = kp people on average

- If p is high
  - The disease will persists through rounds
- If p is low, it will die out after some rounds



(b) *With high contagion probability, the infection spreads widely*



# Property

- When R > 1 number of infected people keeps increasing
  - Outbreak
- When R < 1 Number of infected people decreases
  - Disease dies out
- Phase transition at R = 1
- assuming there are enough “new” people supply to meet
- Generally true in the initial stages

- Around $R = 1$: small efforts can have large effects on epidemic
    - Awareness causing slight decrease in $p$
    - Quarantine/fear causing slight decrease in $k$

# SIR Model

- Susceptible (initially)
- Infectious (after being infected)
  - While Infectious, it can pass disease to each neighbor in each step with prob. p
- Removed (after given duration as Infectious)
  - Immune/dead

# SIS model

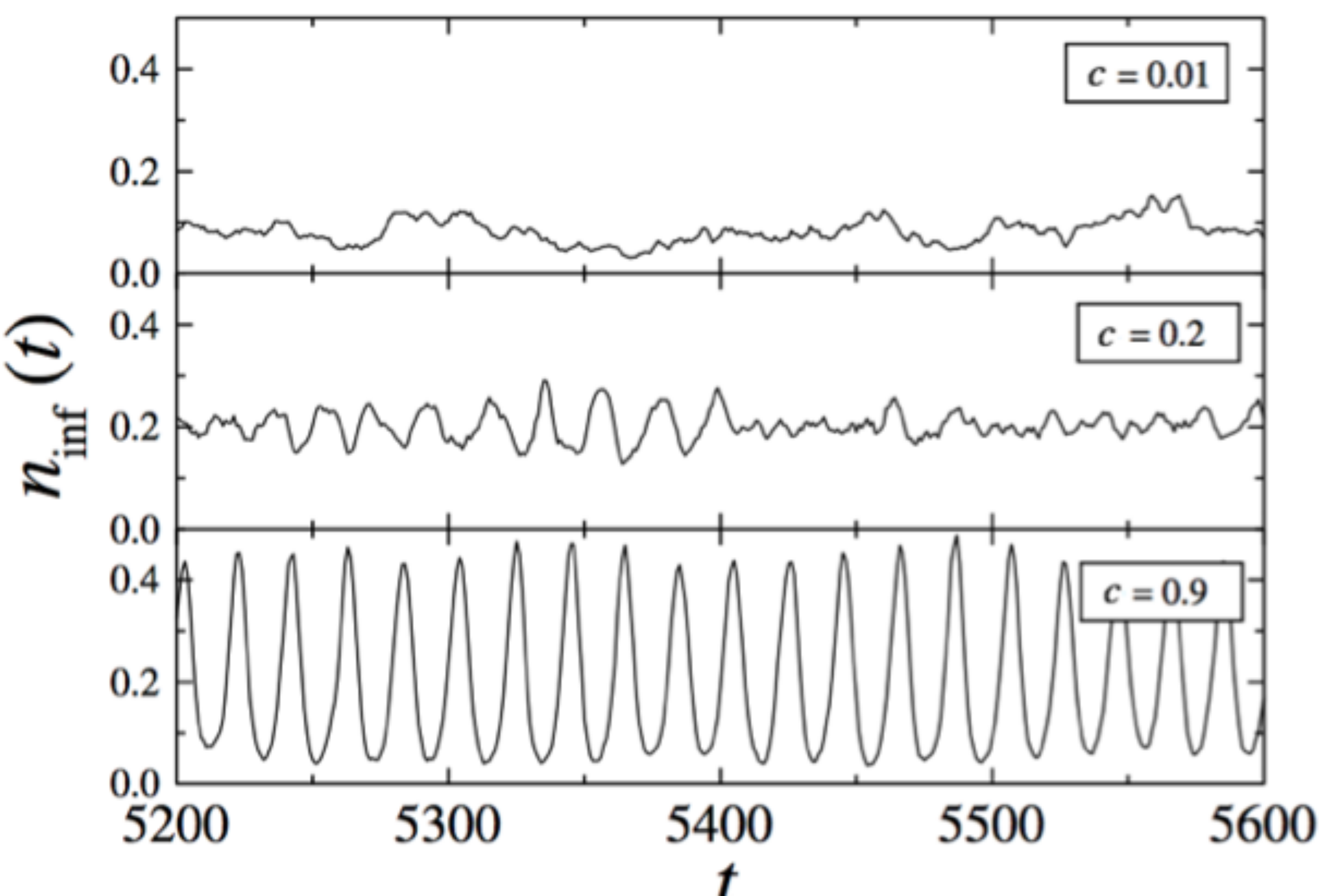
- No “Removed” state. Susceptible follows Infectious
  - Eg. flu

# SIRS model

- Susceptible
- Infectious
- Recovered (immune)
- Susceptible

# SIRS oscillations in Watts-Strogatz Small worlds

- Nodes connected to few nighbors on a ring
- Fraction c of links modified to connect to random nodes



# Epidemic or gossip algorithm

- Emulates the spread of epidemic or a rumor in a network
- A node speaks to a random neighbor to spread the rumor message
- Useful for spreading information in computer networks

# Spreading a message via gossip

- Complete graph: Anyone can call anyone
- In each round, anyone with the rumor calls one node and passes the rumor

# Theorem

- Everyone gets the message in O(log n) rounds
- Idea:
  - n/3 nodes get the message in log n rounds
    - Current number of infected nodes m < n/3
    - Probability that a call goes to a new node is 2/3
      - Number of calls to new nodes: 2m/3
    - Probability of a collision at the new node is 1/(n-m)
    - $O(m^2)$ possible pairs for collision
    - Max possible collisions: $\frac{m^2}{2(n-m)} \leq \frac{m^2}{2} \cdot \frac{1}{2m} = \frac{m}{4}$.
    - Number of newly infected nodes at least $\frac{2m}{3} - \frac{m}{4} = \frac{5m}{12}$

- while m < n/3
  - m grows to m(1 + 5/12) = 17m/12 every round
  - m grows to n/3 in O(log n) rounds
- After m> n/3
  - Probability that a node is not called in 1 round is

$$\left(1 - \frac{1}{n}\right)^{\frac{n}{3}} \leq e^{\frac{-1}{n} \cdot \frac{n}{3}} = e^{-1/3}$$

  - Probability that 1 or more nodes are not called after O(log n) rounds
    - Less thar $1/n^c$, where c depends on the constant in the O

- See Kempe 11, chapter 9.

# Application

- In a computer network (imagine wireless network)
- Spreading a piece of information
- Naive method: A flood: a node calls all its neighbors to send message
  - Wasteful: since many nodes can have common nearby neighbors, this wastes messages
  - Better to do it as one neighbor per round
  - Still spreads slowly…
  - At least $\sqrt{n}$ in a grid of n nodes

# Geographic gossip in wireless networks

- Imagine nodes on a plane
- Instead of only sending messages to neighbors
  - Send to nodes at random (assume there is some routing mechanism in place)
  - Easily reaches far away nodes
  - Faster: O(log n)  rounds: O(n log n) messages
  - but the routing costs more messages:
    - $\sqrt{n}$ hops on average per message
    - $n\sqrt{n}\log n$ total transmissions

# Spread in small world distributions

- Instead send message to a random node
  - Picked according to a small world-like distribution
  - Picking a node at distance d with probability $1/d^3$
  - Note the slight difference in exponent (3 instead of 2)
  - Short paths are more common in these distributions
  - The average message cost is O(poly(log n))
  - Still there are enough long messages spreading the message to far away regions
- Kempe, Kleinberg, Demers; Spatial gossip and resource location protocols STOC 2001

# Advantages of gossip

- Simple to implement
- Robust a lgorithm
  - A node failure does not stop the computation
  - Easy to add nodes to the system
  - At the cost of a logarithmic factor of increased costs

# Averaging gossip

- Suppose the nodes all have a “value”
- And we wish to compute a linear function of these values
- E.g. the average

# Protocols

- The push-sum protocol
  - In every round
  - Every node takes a fraction of its value and sends to a random neighbor
  - It adds all received values to its current value
- The pairwise averaging protocol
  - In every round, a node talks to one other random neighbor
  - Both nodes set their values to the average of the two

# Gossip averaging protocols

- On a complete graph
  - Both protocols converge to the average fast
  - O(log n) rounds
- On small world graphs/small world distributions
  - Convergence not known