Network Curvature:
Structure of the Internet

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Last class: Gossip Protocols

• The push-sum protocol
  • In every round
  • Every node takes a fraction of its value and sends to a random neighbor
  • It adds all received values to its current value

• The pairwise averaging protocol
  • In every round, a node talks to one other random neighbor
  • Both nodes set their values to the average of the two
Gossip averaging protocols

• On a complete graph
  • Both protocols converge to the average fast
  • $O(\log n)$ rounds

• On small world graphs/small world distributions
  • Convergence not known

• On a geometric graph (nodes connected to nearby nodes):
  • Convergence slow

• We will omit proofs.
Internet

• An interconnection network of “network of routers”

• Thousands of networks together form the Internet

• The “center” consists of big routers in highly connected networks, many connections between adjacent networks

• Outer layers have smaller routers and sparser connections
Internet

- Has a layered structure with higher connectivity at the core
  - A routed packet tends to use high connectivity regions to get shorter/faster routes
- Known to have power law distribution of degrees
Metrics

• A metric is a set of (shortest) distances (d) between points

• E.g. distances on a plane satisfy a Euclidean metric

• Distances on a sphere satisfy a spherical metric

• Distances between nodes on a grid graph satisfy a grid metric

• Nodes on a tree satisfy a tree metric
Comparing metrics

- We can say two metric spaces A and B are similar if one can be embedded in the other with small distortion.

- That is, there is a function $f: A \rightarrow B$ such that
  
  $A(x, y) \sim B(f(x), f(y))$
A test for tree metrics

A metric is a tree metric if and only if it satisfies this 4 Point Condition:

• Any 4 nodes (points in the metric space) can be ordered as $w,x,y,z$ such that:

  • $d(w,x) + d(y,z) \leq d(w,y) + d(x,z) \leq d(w,z) + d(x,y)$ and

  • $d(w,y) + d(x,z) = d(w,z) + d(x,y)$
Trees tend to have high loads in “center”

• Since many routes will have to go through the center
Almost tree metrics

- Real networks are not exactly trees
- Let’s measure how far a network is from a tree
- 4PC-$\varepsilon$ for a set of 4 nodes is the smallest $\varepsilon$ that satisfies:
  - $d(w, x) + d(y, z) \leq d(w, y) + d(x, z) \leq d(w, z) + d(x, y)$ and
  - $d(w, z) + d(x, y) \leq d(w, y) + d(x, z) + 2\varepsilon \cdot \min\{d(w, x), d(y, z)\}$
Almost tree metrics

• A tree has $\varepsilon = 0$

• A metric space with smaller $\varepsilon$ implies that it is more similar to a tree

• Theorem: A metric space with small $\varepsilon$ can be embedded into a tree with correspondingly small distortion

• Ref: I Abraham et al. Reconstructing approximate tree metrics, PODC 07.
Treeness of Internet

- PlanetLab: A distributed collection of servers around the world
- Experiment based on latency (communication delay) as an estimate of distance
- Shows the distance metric between servers is similar to a tree, and far from a sphere

- Ref: I Abraham et al. Reconstructing approximate tree metrics, PODC 07.
- V. Ramasubramanian et al. On treeness of internet latency and bandwidth, Sigmetrics 09.
Course and Projects

• More office hours:
  • Thursday 19th, 10:00 - 11:30

• Remaining Classes:
  • Last usual class (some remaining material) Friday 20th
  • No class on Tuesday 24th
  • General review/discussion class on Friday 27th.
Treeness of metrics

- $\delta$-hyperbolic metrics

- $d(w,x) + d(y,z) \leq d(w,y) + d(x,z) \leq d(w,z) + d(x,y)$ and

- $d(w,z) + d(x,y) = d(w,y) + d(x,z) + \delta$

- Uses an absolute value $\delta$
  - Instead of a multiplicative factor
$\delta$-hyperbolic metrics: Thin triangles

- Alternative definition

- Any point on a triangle must be within distance $\delta$ of one of the other sides

- The middle of the triangles are squeezed together

- Trees have $\delta = 0$: most hyperbolic
Curvatures of spaces

- Spherical: +ve curvature
  - Triangle centers are “Fat”
- Flat (Euclidean): 0 Curvature
- Hyperbolic: -ve curvature
Angle properties

• Sum of angles of a triangle
• In Spherical space $\geq \pi$
• In flat space $= \pi$
• In hyperbolic space $\leq \pi$
Growth properties

- Growth of a circle circumference with radius
  - Flat space: grows linearly: $2\pi r$
  - Spherical: grows sub-linearly: $\leq 2\pi r$
    - In fact, as less than $cr$ for any const $c$
    - Shrinks after covering half the sphere
  - Hyperbolic: Grows super-linearly: $\geq 2\pi r$
    - In fact, exponentially as $\sinh(r) \sim r^e$
    - Does not fit on the plane or even 3D
• Any hyperbolic space is $\delta$-hyperbolic for some finite delta

• Not the case for Euclidean and spherical spaces

• For more on $\delta$-hyperbolic spaces, See: Gromov hypoerbolic spaces
• Any tree can be embedded in a hyperbolic space with a low distortion

Internet

- Internet has good embedding in hyperbolic spaces

Modeling metrics

- We need a set of points
- A notion of “straight line distances”

- On euclidean plane, straight lines are straight lines
- On a sphere, straight lines are “great circles”
A model for hyperbolic metrics

- The Poincare model
  - Take a disk
  - Straight lines are circular arcs that meet the boundary at right angles
  - Note that there can be many more parallel lines than in flat space
The internet structure

- Observe:
  - Shortest paths tend to come close to the center and then move away
  - Explains “internet core”
Visualizing hyperbolic metrics

- All elements are same size
- Things close to the boundary “look” small
  - Perspective
- When drawing an infinite surface, some things have to be compressed
- See Escher’s paintings
A model for hyperbolic metrics