Social and Technological Networks	Lecture 0

Sample problems

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The following are sample problems to test your background for the course. You should be able to solve most of these without looking up references (or, without looking up too many of them). At least solve the easy ones, and you should have some idea of what the harder ones mean and how to approach them. Also see the exercise problems in the lecture notes for Lecture 1 as examples of types of questions in the course.
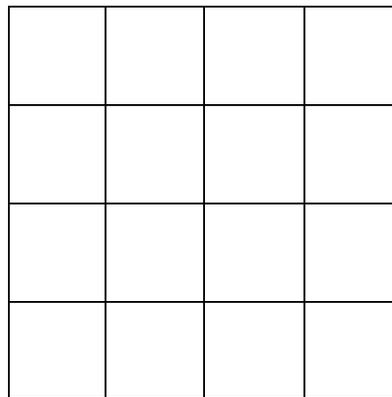
Figure 1: Example of a Grid.

Exercise 1 *A grid is an arrangement of squares as shown in Fig. 1. Prove that for any given grid, the number of grid squares inside a circle of radius r is $O(r^2)$.*

Exercise 2 *Show that a bipartite graph has no cycles of odd length.*

Exercise 3 *An* isolated *vertex is one which has no edges. Consider a graph G with n vertices such that every edge exists with probability $p = (1+\varepsilon)(\ln n)/(n-1)$. Prove that the probability that G has one or more isolated vertices is less than $1/n^\varepsilon$.*
   [Hint: Write the probability that none of the possible edges at a vertex exist. Use the inequality $(1-p)^{1/p} \le 1/e$ for $0 \le p \le 1$. You can also use the Union bound, which says $Pr[A \text{ OR } B] \le Pr[A] + Pr[B]$.]

Harder problems:

Exercise 4 *Show that the matrix $M = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$ has orthogonal eigenvectors for any real numbers a, b. [Hint: Try comparing values of $(Mv) \cdot u$ and $(Mu) \cdot v$ for vectors u and v, then use definition of eigen vectors. You can use the fact that M has eigen values λ and μ that are distinct.]*

Exercise 5 *Let us define matrices A and B to be* similar *if there exists a matrix P such that* $A = PBP^{-1}$.

For similar matrices A and B, show that if λ is an eigenvalue of A, then it is also an eigenvalue of B. [Hint: Use definition of eigen vector, then multiply both sides by suitable matrices. The eigen vectors corresponding to the eigen value may not be the same. You can assume A, B, P are square.]