Social and Technological Networks Lecture 0

Sample problems

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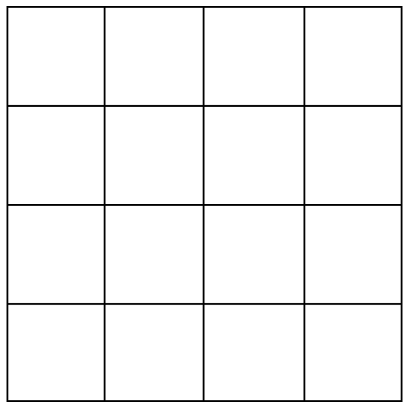

Figure 1: Example of a Grid.

Exercise 1 *A grid is an arrangement of squares as shown in Fig. 1. Prove that for any given grid, the number of grid squares inside a circle of radius r is $O(r^2)$.*

Proof Let us suppose each grid square has side s, and area s^2. Since the interiors of the grid squares are disjoint, the total area covered by any n distinct grid squares is ns^2. The area of the circle of radius r is πr^2, and the maximum number of possible squares in the circle is $\leq \pi r^2/s^2$. For a given grid s is fixed, so the number of squares in the circle is $O(r^2)$. $\square$

Exercise 2 *Show that a bipartite graph has no cycles of odd length.*

Proof Suppose the two partitions are U and V. Without loss of generality, let us suppose that the cycle C starts from $u \in U$. By definition of a bipartite graph, traversal along C must alternate between the $U \to V$ type on odd numbered edges and the $V \to U$ type on even numbered edges. Since the cycle must end at $u \in U$, it must end with a $V \to U$ type edge which is even numbered. Thus C must have even numbered edges. $\square$

Exercise 3 *An* isolated *vertex is one which has no edges. Consider a graph G with n vertices such that every edge exists with probability $p = (1+\varepsilon)(\ln n)/(n-1)$. Prove that the probability that G has one or more isolated vertices is less than $1/n^{\varepsilon}$.*

[Hint: Write the probability that none of the possible edges at a vertex exist. Use the inequality $(1-p)^{1/p} \leq 1/e$ for $0 \leq p \leq 1$. You can also use the Union bound, which says $Pr[A \text{ OR } B] \leq Pr[A] + Pr[B]$.]

Proof At vertex v, probability that a particular edge does not exist is $(1-p)$; the probability q that the vertex v is isolated, i.e. all $n-1$ possible edges do not exist is $q=(1-p)^{n-1}$. We can substitute $n-1=(1+\varepsilon)(\ln n)/p$ in the exponent, and get $q \leq e^{-((1+\varepsilon)(\ln n))}$. Therefore, $q \leq n^{-(1+\varepsilon)}$.

By union bound, the probability that over n vertices, one or more is isolated is $\leq nq \leq n^{-\varepsilon}$. □

Exercise 4 *Show that the matrix* $M=\begin{pmatrix} a & b \\ b & a \end{pmatrix}$ *has orthogonal eigenvectors for any real numbers* a, b. *[Hint: Try comparing values of* $(Mv)\cdot u$ *and* $(Mu)\cdot v$ *for vectors* u *and* v*, then use definition of eigen vectors. You can use the fact that* M *has eigen values* λ *and* μ *that are distinct.]*

Proof $(Mv)\cdot u=\begin{pmatrix} av_1+bv_2 \\ bv_1+av_2 \end{pmatrix}\cdot\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}=av_1u_1+bv_2u_1+bv_1u_2+av_2u_2.$

And $(Mu)\cdot v=\begin{pmatrix} au_1+bu_2 \\ bu_1+au_2 \end{pmatrix}\cdot\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}=au_1v_1+bu_2v_1+bu_1v_2+au_2v_2.$

Thus $(Mv)\cdot u=(Mu)\cdot v$ for any vectors u and v. Now suppose u and v are eigen vectors of M, with eigen values λ and μ. Then $(\lambda u)\cdot v=(Mu)\cdot v=(Mv)\cdot u=(\mu v)\cdot u$.

Since $\lambda \neq \mu$, it follows that $(u\cdot v)(\lambda-\mu)=0$ implies $u\cdot v=0$, that is, u and v are orthogonal. □

Exercise 5 *Let us define matrices* A *and* B *to be* similar *if there exists a matrix* P *such that* $A=PBP^{-1}$.

For similar matrices A *and* B*, show that if* λ *is an eigenvalue of* A*, then it is also an eigenvalue of* B*. [Hint: Use definition of eigen vector, then multiply both sides by suitable matrices. The eigen vectors corresponding to the eigen value may not be the same. You can assume* A, B, P *are square.]*

Proof Let x be an eigen vector of A with eigen value λ. Also, let use denote $P^{-1}x=y$.

$$
\begin{aligned}
Ax &= \lambda x \\
\Rightarrow PBP^{-1}x &= \lambda x \\
\Rightarrow BP^{-1}x &= P^{-1}\lambda x = \lambda P^{-1}x \text{ [After Left-multiplication by } P^{-1}\text{]} \\
\Rightarrow By &= \lambda y \text{ [Substituting } y\text{.]}
\end{aligned}
$$

Therefore, λ is also an eigen value of B with eigen vector y. □