Social and Technological Networks	Edinburgh, 2015

Lecture 9. Spectral graph theory.

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We looked at the following:

- Basic ideas in spectral graph theory: Laplacian
- Embedding or graph drawing
- Graph Coloring
- Graph Segmentation and clustering
- Image segmentation

A small note on the topic of embedding.

Embedding usually refers to mapping one object (like a graph) to another object or space (like a plane, 3D space) in a way that preserves the topology (connectivity properties) of the original object. *Drawing* usually refers to an embedding in the plane.

Mathematically, an embedding is a map of an object X into a space Y, written as $f : X \to Y$, and to say that a map is an embedding usually implies that no two points of X map to the same point in Y. Note that an embedding need not always exist. For example, a plane cannot be embedded into a line, though a line can be embedded in a plane.

In case of graphs, we ordinarily take this to mean that no two vertices map to the same point. The space Y may be $2D$, $3D$ or other even other bent spaces like sphere, torus etc. Y can even be a graph.

In some cases, particularly of embedding in $2D$, we may require that edges do not cross each-other. For example, in drawing some types of maps

Image segmentation. This works by creating a grid graph overlayed on the image and assigning weights to the edges. The weight is the similarity in the pixel values of the two end-points. For edge (i, j), $weight(i, j) \approx e^{-(px_i - px_j)^2}$, where px_i is the pixel value at i.[1]. This expression for weight means that neighboring vertices with identical values have the highest possible weight $= 1$ (thick edges in the picture on the slide), while edges with dissimilar end-points have smaller weights (thinner edges in the picture).

The weights change the matrices: A will contain the weights instead of just a 1 for an edge, and D the total weight of edges at the vertex. L will change accordingly to represent $L = D - A$.

As a result, in the heat flow interpretation, an edge between similar vertices has greater *conductivity*. Of the heat at a vertex (say a blue vertex), more heat is likely to flow to a similar (blue) vertex.

[1] The slides presented in class had a typographic error which has been corrected

(In a random walk interpretation, this simply means that a random walk is more likely to move to a similar vertex.)

This means, that nearby vertices with similar pixel values are likely to have similar heat levels after a small number of steps. This fact is reflected in the first few eigen vectors (such as $v[1]$) these sets of pixels have similar values.

Laplacian What we have seen till now is a discrete version of the Laplacian, written as a difference equation for a discrete graph. Classically, the Laplacian is a smooth differential operator working in a d-dimensional space:

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_d^2},$$

Where $x_1, \ldots x_k$ are the coordinates corresponding to the d dimensions for the space. This operator works on functions that assign a value to each point in the space. For example, this function u can be the heat content at each infinitesimally small element of the space (temperature). Thus the definition of laplacian basically tells us to sum the second derivative of u along each coordinate:

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_d^2}.$$

Note that like u, Δu also produces a single value at each point in the space. As in the case of graphs, the graph Laplcian operates on a vector to produce another vector with one value for each node.

In case of a $1D$ continuous space space, Δu will simply be the second derivative of u at each point.

Exercise 0.1. *Suppose $u = \sin(x)$, what is Δu?*

One way to think of the analogy between the graph laplacian and the smooth laplacian to consider the grid approximation of the euclidean space (as in the segmentation picture) and to say that the smooth case arises as the limit of the discrete case, with the grid becoming infinitely fine. However, we will omit this for the course.

Heat equation. Suppose we put a small and different quantity of heat at each node, and let that diffuse. What is the rate of change with time, of heat content at each node, or each point of space? Clearly, this should be written as $\frac{\partial u}{\partial t}$, where t is time. Both u and t can be either continuous or discrete. The important equation is that the rate of change equals the laplacian:

$$\frac{\partial u}{\partial t} = c\Delta u.$$

Where c is some suitable constant. That is, how much the heat at a point increases or decreases, depends on the second derivative with respect to heat levels at neighboring points. Another way to think of it is that this change depends on the gradient of gradients with neighboring points (hence the second derivative). That there should be some such relation to levels at neighboring points is intuitive, but it is quite nice the rate of change fits perfectly proportionally with the second derivative.

We will not derive the heat equation. However, in the discrete case, it is not that hard to derive. You can try for yourself.

* **Exercise 0.2.** *Imagine a metal rod with clear markings that separate it into small pieces of equal size. Now imagine that we heat up different pieces to different temperatures. Consider time in small equal sized intervals (such as 1ms) over which the temperatures stay approximately constant. Derive the flow of heat across each marking per time unit and hence the discrete heat equation.*

Now consider a plane divided similarly into a grid and derive the $2D$ *discrete heat equation.*