Social and Technological Networks Edinburg, 2015

Lecture 1. Basics and sample problems

TA: Maria Astefanoaei *Solutions to Exercises*

Exercise 1. *How many edges can a graph have? (assuming there is at most one edge between any two vertices.) If each possible edge exists with a probability p, what should be the value of p such that the expected number of edges at each vertex is 1?*

Answer. Assuming it is a simple graph, there is at most one edge between any pair of nodes. And there are $\binom{n}{2}$ nodes. Thus a grpah can have $\binom{n}{2} = \frac{n(n-1)}{2}$ edges.

A node v can have at most $n-1$ edges incident on it. Each of these exists with a probability p independent of the others. The expected number of edges at node v is $(n-1)p$. Therefore we can solve for p from $(n-1)p = 1$, therefore $p = \frac{1}{n-1}$.

Exercise 2. *Suppose every year Mr. X makes double the number of friends he made last year (starting with making 1 friend in first year). In how many years will he make n friends? (asymptotic notation is fine.)*

Answer. Mr. X makes 1 friend in the first year, 2 in the second year, so he has in total $1+2$ friends in the second year. At the end of m-th year he will have $1+2+\ldots 2^{m-1} = 2^m - 1$ friends. Now let us select the smallest m such that $2^m - 1 \geq n$. Observe that by this definition, after year $m-1$, he had strictly less than n friends, and after year m he can actually have much more than n friends. However, m is still the right answer, because we are counting whole years.

Expressing m in terms of n, we have $m = \lceil \lg(n+1) \rceil$ [1]. We have to use the ceiling function here because $n+1$ may not be a power of 2, and we need to take the next integer to get a proper count.

Exercise 3. *Suppose we throw k balls into n bins randomly, what is the probability that bin 1 remains empty?*

Answer. $\Pr[\text{bin 1 is empty after 1 throw}] = 1 - \frac{1}{n}$. Therefore, $\Pr[\text{bin 1 is empty after k throw}] = (1 - \frac{1}{n})^k$.

Exercise 4. *Show that for a unit grid in a plane (as above), $|v_1 - v_2|_1 = \Theta(|v_1 - v_2|_2)$, for any v_1, v_2 in the graph.*

[1] The $\lceil \bullet \rceil$ symbol stands for the function *ceiling* implying the integer greater than or equal to its argument.

Proof: To prove $|v_1 - v_2|_1 = \Theta(|v_1 - v_2|_2)$ we need to find constants c_1 and c_2 such that

$$c_1|v_1 - v_2|_2 \leq |v_1 - v_2|_1 \leq c_2|v_1 - v_2|_2,$$

which is equivalent to

$$c_1\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \leq |x_1 - x_2| + |y_1 - y_2| \leq c_2\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Raising everything to the power 2 we get

$$c_1^2((x_1-x_2)^2+(y_1-y_2)^2) \leq (x_1-x_2)^2+(y_1-y_2)^2+2|x_1-x_2||y_1-y_2| \leq c_2^2((x_1-x_2)^2+(y_1-y_2)^2).$$

By subtracting $(x_1 - x_2)^2 + (y_1 - y_2)^2$ the inequalities become

$$(c_1^2 - 1)((x_1 - x_2)^2 + (y_1 - y_2)^2) \leq 2|x_1 - x_2||y_1 - y_2| \leq (c_2^2 - 1)((x_1 - x_2)^2 + (y_1 - y_2)^2).$$

The first inequality is

$$(c_1^2 - 1)((x_1 - x_2)^2 + (y_1 - y_2)^2) \leq 2|x_1 - x_2||y_1 - y_2|$$

and it is easy to see that if $c^2 - 1 = 0$, so if $c_1 = 1$ it is satisfied.

For the second inequality we can use that $2ab \leq a^2 + b^2$ for any real a, b. Therefore $c_2^2 - 1 = 1$ satisfies the inequality, so $c_2 = \sqrt{2}$.

To conclude, we have that

$$|v_1 - v_2|_2 \leq |v_1 - v_2|_1 \leq \sqrt{2}|v_1 - v_2|_2,$$

which means that $|v_1 - v_2|_1 = \Theta(|v_1 - v_2|_2)$.

[Comments: 1. There is a geometric way of proving this. Try it yourself. 2. The constants 1 and $\sqrt{2}$ derived above are exact, but there is a slightly less complex way to prove for constants 1 and 2.] □

Exercise 5. *Show that for a unit grid,* $|B_r(v)| = O(r^2)$.

Proof: The number of vertices inside a circle of radius r cannot be more than the number of vertices inside a grid $(2r+1) \times (2r+1)$, which has $(2r+1)^2$ vertices. Therefore, $|B_r(v)| \leq (2r+1)^2 = 4r^2 + 4r + 1 = O(r^2)$. □