# Structure and Synthesis of Robot Motion

## Introduction: Some Concepts in Dynamics and Control

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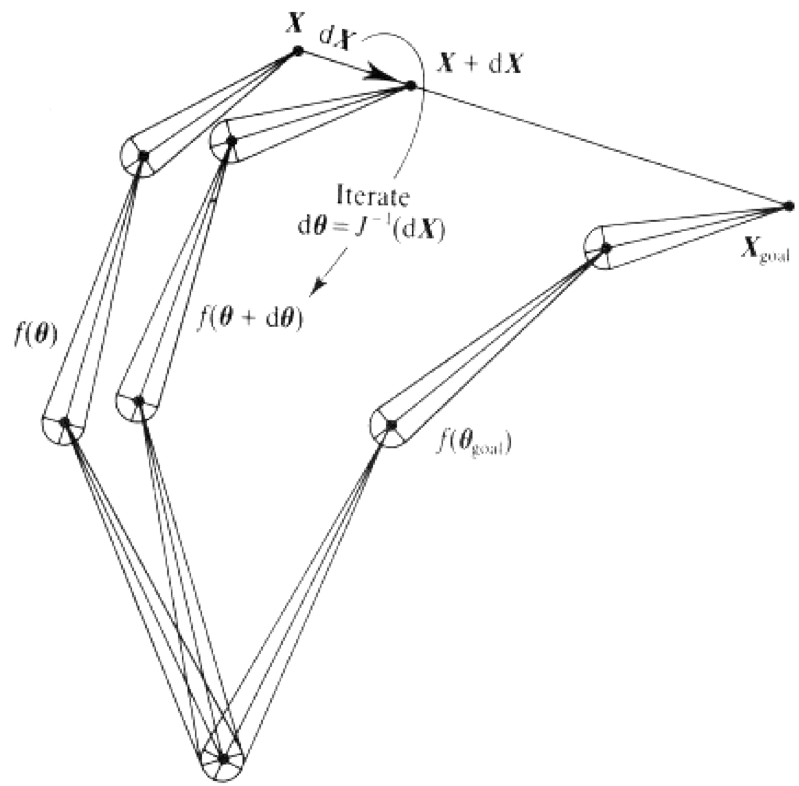
# How do we describe motion?

From the previous lecture:

- Describe what needs to happen in the task space (e.g., trace a line)
- Use kinematics calculations to figure out what this means for joints (c-space)




Next question:

Dynamics: **when** should each c-space point be visited?

# Motion of a Particle

Consider the case of a single particle of mass $m$ that moves in a world $W = \mathbb{R}$.

The applied force is a scalar $f \in \mathbb{R}$.

Let $q$ denote the state (position) of the particle in $W$ at time $t$.

Then, $\ddot{q} = \frac{f}{m}$

If you fix mass, say $m = 1$, and force, say $\ddot{q} = u$, then the motion is completely described by two variables:

$\dot{x_1} = x_2$

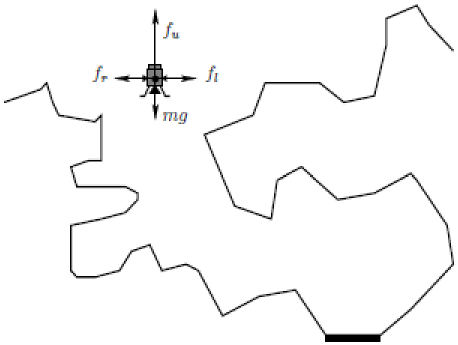
$\dot{x_2} = \frac{u}{m}$

# Motion of a Lunar Lander

There are three thrusters and gravity acts on the vehicle. We ignore rotations.

We model this as a particle of mass $m$ in a 2-dim world.

We can write equations of motion as ($u_i \in [0, 1]$),




$$m\ddot{q}_1 = u_l f_l - u_r f_r$$

$$m\ddot{q}_2 = u_u f_u - mg$$

Using 2 positions and velocities, we could write this as 4 first-order equations.

# How to Move Beyond Point Masses?

- Clearly, we want to be able to model more complex robots as they appear in practice
- Within the Newton-Euler formalism, one can try to derive further conservation laws, e.g., momentum
- This yields additional equations that act as differential constraints on the state space

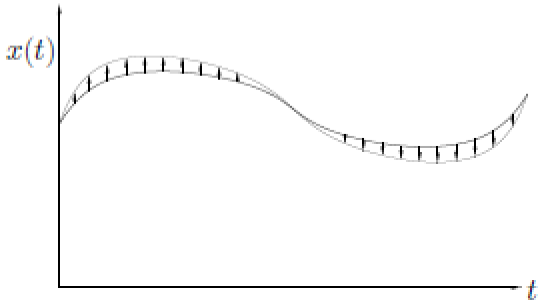
# One 'Modern'/General Method: Lagrangian Mechanics

- Based on calculus of variations – optimisation over the space of paths
- Motion is described in terms of the minimisation of an action functional:

$$\Phi(x) = \int_T L(x(t), \dot{x}(t), t)dt$$

Example: Shortest-path functional

$$L(x, \dot{x}, t) = \sqrt{1 + \dot{x}^2}$$




Optimization is over possible **small perturbations in functional form**

# Recap from Calculus: Extrema of Functions

**Extreme Value Theorem**

Suppose $f(x)$ is continuous on interval $[a, b]$ then there are two numbers $a \leq c, d \leq b$ such that $f(c)$ is an absolute maximum and $f(d)$ is an absolute minimum for the function.

**Fermat's Theorem**

If $f(x)$ has relative extrema at $x = c$ and $f'(c)$ exists then $x = c$ is a critical point of $f(x)$. In fact it will be a critical point such that $f'(c) = 0$

So, $f(x)$ is a continuous function on the interval $[a, b]$,

- Find critical points by solving $f'(x) = 0$
- Check which ones are absolute maxima/minima

Confirming maxima or minima: Second Derivative
$f''(x) > 0$: minima
$f''(x) < 0$: maxima

# Similar Question for *Paths*

Consider all possible paths joining two points A and B in the plane. Suppose a particle is allowed to move along any of these paths and let the particle have a definite velocity $v(x, y)$ at each point $(x, y)$. Then one can define a functional associating each path with the time taken.

Based on this we can ask about absolute maxima/minima for paths (shortest time).

We seek $y(x)$ so that $J = \int_a^b \sqrt{(1 + \dot{y}^2)}$ is minimized.

# An Interesting Question about Paths

*Brachistochrone* problem (Johann Bernoulli, 1696):

Let $P_1 = (0, 0)$ and $P_2 = (1, -1)$ be two points in the plane. Connect them by a differentiable curve $y = f(x)$. Suppose a bead is allowed to slide, frictionless, along these curves then what is the curve that minimizes the time taken by the bead?

Bead's motion: $y(t) = f(x(t))$

$v_1(t) = \dot{x}(t)$ and $v_2(t) = f'(x(t))\dot{x}(t)$

$|v(t)| = \sqrt{1 + [f'(x(t))]^2}\dot{x}(t)$

Conservation of energy: $\frac{1}{2}mv^2 = -mgy(t)$

$\frac{\sqrt{1+[f'(x(t))]^2}}{\sqrt{-2gf(x(t))}}\dot{x}(t) = 1$

$T = \int_0^1 \frac{\sqrt{1+[f'(x(t))]^2}}{\sqrt{-2gf(x(t))}}dx$

This is to be minimized, over all **paths** $f(x(t))$.

# *Q.*: What is the Derivative of a Functional?

- From basic calculus, we know what to do with functions:

$$y = f(x); \;\; x, y \in \Re$$

$$f'(x) = \frac{dy}{dx}$$

  – A statement about how $y$ changes for "small" changes in $x$

- But now, instead of $x$ & $y$, we have: $T = \int_0^1 \frac{\sqrt{1+[f'(x(t))]^2}}{\sqrt{-2gf(x(t))}} dx$

- How does $T$ change with respect to "small" changes in $f(x(t))$
  – What does it mean to make a "small" change in $f(x(t))$?

# Euler-Lagrange Equation

*Differentiability of a functional*:

If you have a functional $I : \chi \mapsto \mathbb{R}$, it is differentiable at $f$ if there is a continuous linear functional $T : \chi \mapsto \mathbb{R}$ such that $\forall g \in \chi$,

$$T[g] = \lim_{t \to 0} \frac{I[f + tg] - I[f]}{t}$$

If $I[y] = \int_a^b L(x, y(x), \dot{y}(x))dx$ then $\forall z \in \chi$,

**Through a derivation to work out when $\delta I$ (term within limits) vanishes**

$$\boxed{\frac{\partial L}{\partial y} - \frac{d}{dx}(\frac{\partial L}{\partial y'}) = 0}$$

# Solving the *Brachistochrone* Problem

Functional is defined in terms of:

$$L = \frac{\sqrt{1 + [f'(x(t))]^2}}{\sqrt{-2gf(x(t))}}$$

With $y = f(x(t))$, apply

$$\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = 0$$

The solution is a family of *cycloids*:

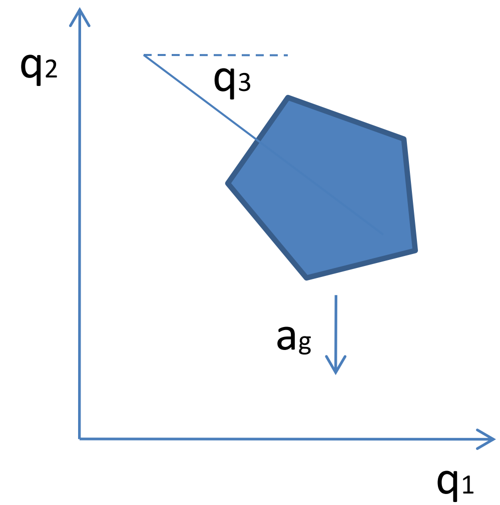
$$x = r(\theta - \sin\theta) + c$$
$$y = r(1 - \cos\theta)$$

A ball will roll down the cycloid faster than the straight line!

# Deriving Equations of Motion from Lagrangians: A Planar Robot

Begin with setting up the Lagrangian:

$$L = \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2) + \frac{1}{2}I\dot{q}_3^2 - ma_g q_2$$




$$\begin{aligned}
u_1 &= \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} - \frac{\partial L}{\partial q_1} = \frac{d}{dt}(m\dot{q}_1) - 0 = m\ddot{q}_1 \\
u_2 &= \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2} - \frac{\partial L}{\partial q_2} = \frac{d}{dt}(m\dot{q}_2) - ma_g = m\ddot{q}_2 - ma_g \\
u_3 &= \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_3} - \frac{\partial L}{\partial q_3} = \frac{d}{dt}(I\dot{q}_3) - 0 = I\ddot{q}_3
\end{aligned}$$

External forces balance accel. terms

# RP Manipulator

This is a more complex example, so we need to write each term separately.

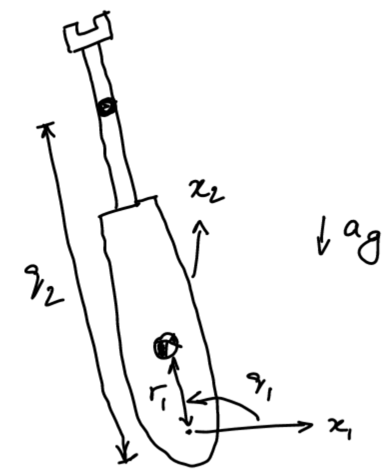
KE of first-link, $K_1(q, \dot{q}) = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} I_1 \omega_1^2$

PE of first-link, $V_1(q) = m_1 a_g r_1 \sin q_1$

KE of second-link, $K_2(q, \dot{q}) = \frac{1}{2} m_2 v_2^2 + \frac{1}{2} I_2 \omega_2^2$

PE of first-link, $V_2(q) = m_2 a_g q_2 \sin q_1$

The complete Lagrangian is $L = K_1 + K_2 - V_1 - V_2$




$$L = \frac{1}{2}((I_1 + I_2 + m_1 r_1^2 + m_2 q_2^2)\dot{q}_1^2 + m_2 \dot{q}_2^2) - a_g \sin q_1 (m_1 r_1 + m_2 q_2)$$

# RP Manipulator Equations

The complete Lagrangian is $L = K_1 + K_2 - V_1 - V_2$

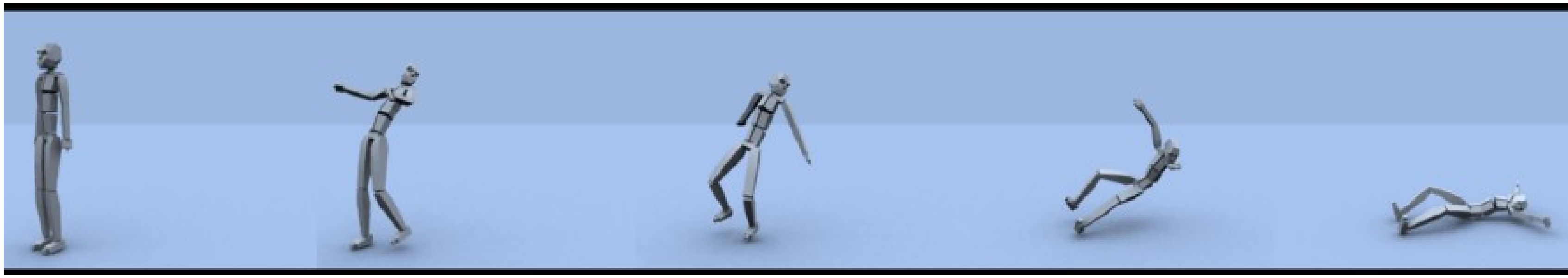
$$L = \frac{1}{2}((I_1 + I_2 + m_1 r_1^2 + m_2 q_2^2)\dot{q}_1^2 + m_2 \dot{q}_2^2) - a_g \sin q_1 (m_1 r_1 + m_2 q_2)$$

From this, we can extract equations of motion as,

$$u_1 = (I_1 + I_2 + m_1 r_1^2 + m_2 q_2^2)\ddot{q}_1 + 2 m_2 q_2 \dot{q}_1 \dot{q}_2 + a_g (m_1 r_1 + m_2 q_2) \cos q_1$$

$$u_2 = m_2 \ddot{q}_2 - m_2 q_2 \dot{q}_1^2 + a_g m_2 \sin q_1$$

# Question



[Source: NaturalMotion Ltd.]

- The equations of motion (and corresponding geometric/ kinematic descriptions) will tell you what will happen given initial conditions, forces, etc.
- You want exactly the opposite – what should you tell the robot so it will go through the states you are interested in?!

# The Optimal Control Problem

- Given a dynamical system with **states** and **controls**
- Find **policy** or sequence of control actions $u(t)$ up to some final time
- Forcing the state to go from an initial value to a final value
- While minimizing a specified cost function

The resulting state trajectory $x(t)$ is an **optimal trajectory**

Remarks:

- Certain combinations of cost functions and dynamical systems yield analytical solutions (most need to be solved numerically…)
- Often, the control policy can be described as a feedback function or *control law*

# The Optimal Control Problem

We are looking for an optimal control $u^*(t)$ $t_0 \leq t \leq t_f$ that minimizes

$$J = \phi[x(t_f), w(t_f), p(t_f), t_f] + \int_{t_0}^{t_f} L[x(t), u(t), w(t), p(t), t]dt$$

where $x$ is the state, $u$ is the control, $w$ are disturbances, $p$ are physical parameters. Often, $w$ and $p$ are fixed and/or known.

So, the cost becomes,

$$J = \phi[x(t_f), t_f] + \int_{t_0}^{t_f} L[x(t), u(t), t]dt$$

Note that there are two distinct terms to the cost - a final term and a running term.

# The Optimal Control Problem

This optimal control, $u^*(t)\ t_0 \leq t \leq t_f$, acts on a dynamic system

$$\dot{x}(t) = f[x(t), u(t), t]$$

whose solution trajectory is,

$$x(t) = x_0 + \int_{t_0}^{t_f} f[x(\tau), u(\tau), \tau] d\tau$$

Often, one uses a general *quadratic* cost function in applications,

$$J = \int_{t_0}^{t_f} \left\{ (x'(t) u'(t)) \begin{pmatrix} Q & M \\ M' & R \end{pmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \right\} dt$$

$$J = \int_{t_0}^{t_f} x'(t) Q x(t) + 2x'(t) M u'(t) + u'(t) R u(t) dt$$

Such a cost function would penalize control effort and deviation of the distance from the desired state. The final cost term acts as a *soft* constraint on the end-point goals.

# Classical Solution: Necessary Conditions for Optimality

The solution to the optimal control problem can be captured by three necessary conditions:

$$
\begin{aligned}
\dot{\lambda}'(t) &= \{-\frac{\partial H[\cdot]}{\partial x}\}' \; t_0 \leq t \leq t_f \\
\lambda(t_f) &= \{\frac{\partial \phi[\cdot]}{\partial x}\}' \\
\{\frac{\partial H[\cdot]}{\partial u}\} &= \frac{\partial L}{\partial u} + \lambda' \frac{\partial f}{\partial u}
\end{aligned}
$$

where $H[x(t), u(t), \lambda(t), t] = L[x(t), u(t), t] + \lambda'(t) f[x(t), u(t), t]$

# Some Observations

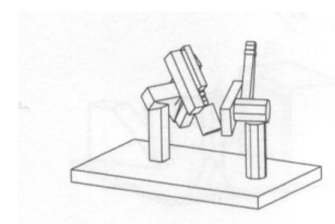
- In robotics, the variability across tasks is exceptionally large
  - Robot morphologies differ
  - Task specifications differ
  - Disturbance classes and sources of uncertainty differ
- We do have certain coarse classification of types of tasks – manipulation, locomotion, area coverage, etc.
- But we are very far from being able to standardize our applications down to modules that can be entirely abstracted and dealt with using traditional program design approaches

- Let us look at one single (relatively old, but instructive) system to understand what the needs are …

# Handey

- System created by Thomas Lozano-Perez and many collaborators at MIT AI Lab
- Described in a book: T. Lozano-Perez, J.L. Jones, E. Mazer, P.A. O'Donnell, *Handey: A Robot Task Planner,* MIT Press, 1992.
- Goal was to explore what it takes to program pick-and-place robots with certain basic properties:

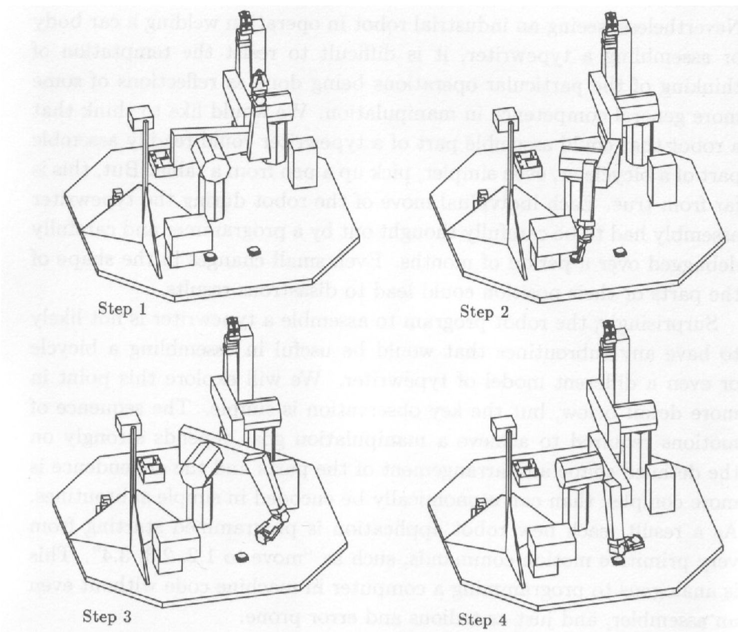
> The word robot should conjure up the image of a system with (at least) three generic capabilities:
>
> - The ability to perceive its environment and to locate objects of interest.
> - The ability to act on its environment.
> - The ability to plan actions to achieve its goals.



# State of the Art, in 1992

- When systems like Handey were being built, robots were being programmed at a very primitive level (“move to 1.2, 2.3, 3.4”) – like very primitive machine code
- Now, programming a pick-and-place task for a specific object, in a specific environment, with a specific robot, and to a specific destination is not that difficult.
  - The difficulty resides in achieving some degree of generality (i.e., autonomy)
- Research Goal: Manipulate a wide class of objects in a wide class of environments using a wide class of robots

# What Handey Could Do

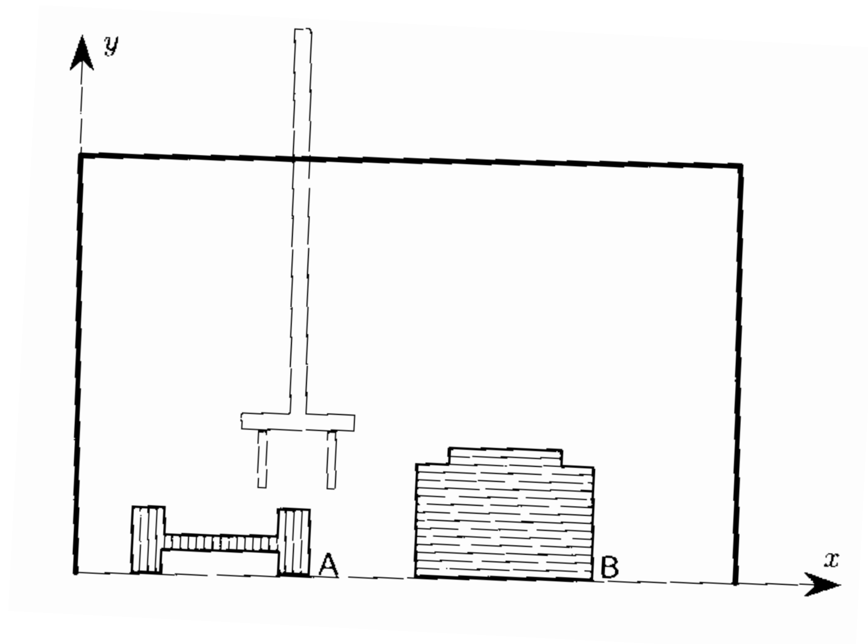



- Input: Object models for parts to be manipulated; High-level Move commands involving objects and destinations
- What Robot Does: Locates parts, picks a grasp, plans a path and moves towards goals (in cluttered environment)

# What Problems might the Robot Face? Unexpected Changes

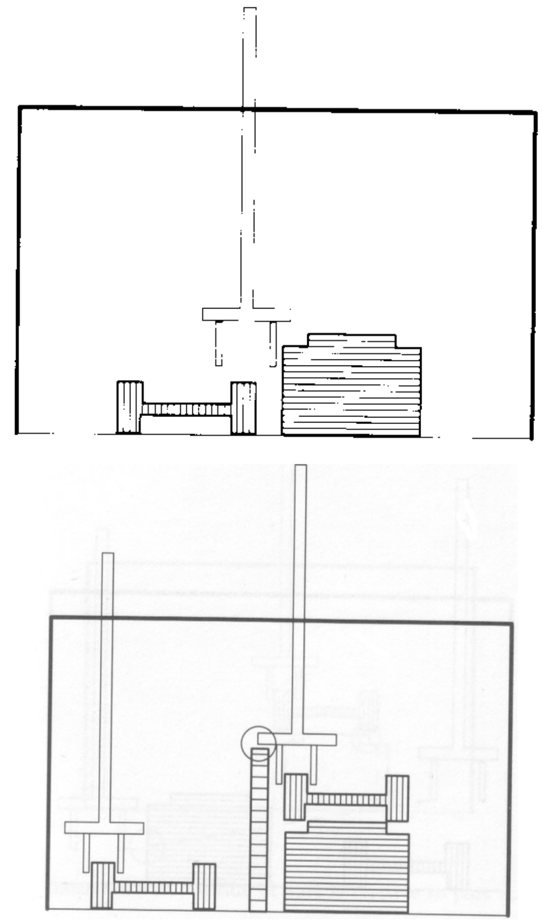
Consider the simplest issue:

- If you pre-program movements then what happens if A or B are not exactly where you thought they would be?
- In cases where you have sensory feedback, how might you account for still larger scale unexpected events (e.g., missing B)?



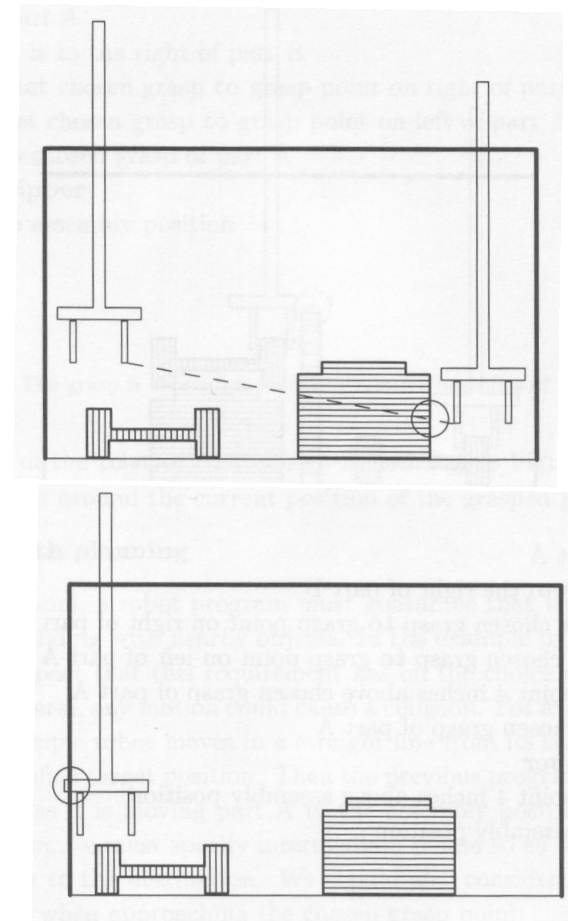
# What Problems might the Robot Face? Workspace Obstacles

If you pre-program *movement patterns (e.g., ways to grasp)* then what happens when obstacles impede your path?



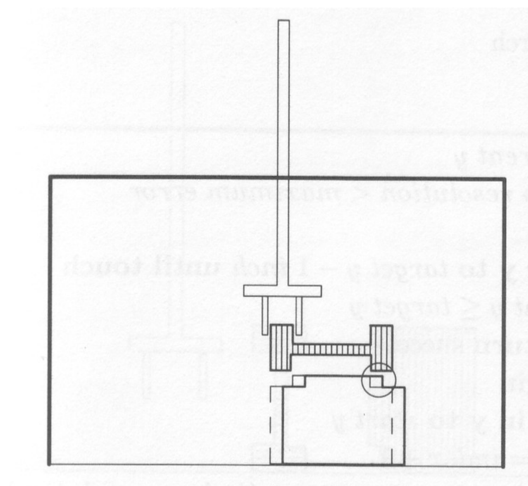
# What Problems might the Robot Face? Computing Collision-Free Paths

How to compute the gross motion of getting from start pose to destination pose, respecting all constraints along the way?



# What Problems might the Robot Face? Uncertainty

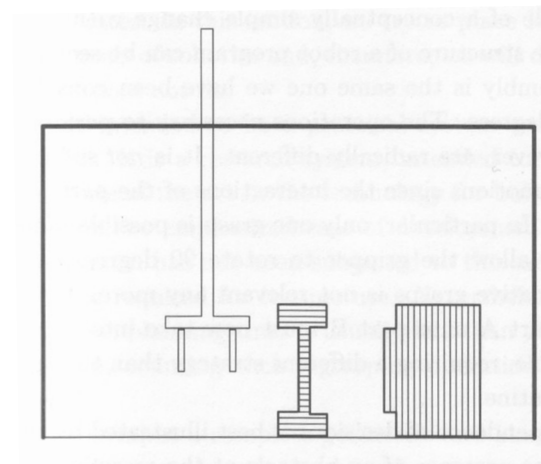
How to compute the gross/fine motion of the robot and objects, when none of the coordinates are known exactly (e.g., due to sensor/motor noise)?



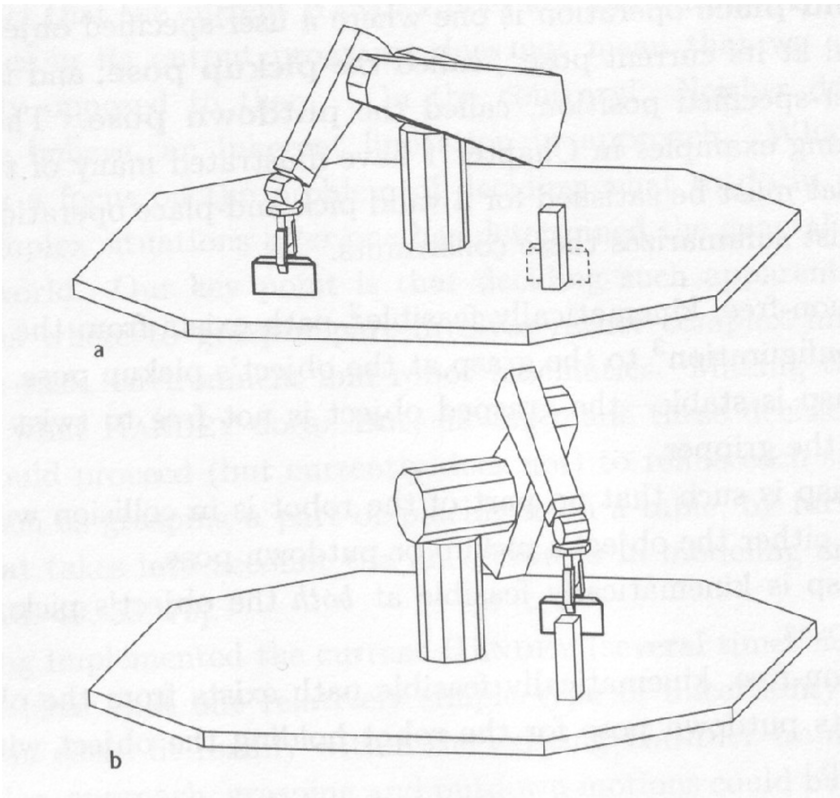
# What Problems might the Robot Face? Error Detection and Recovery

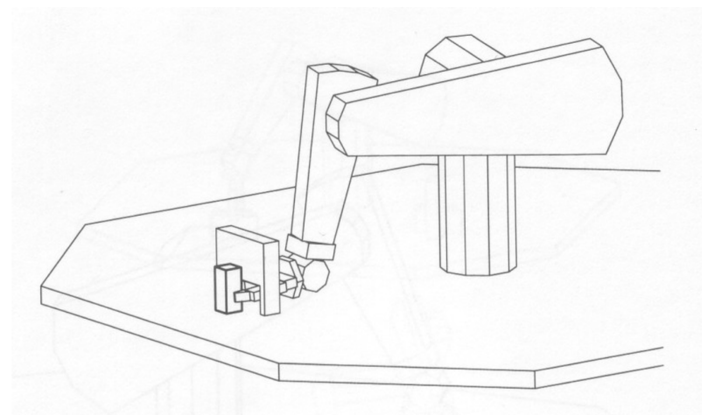
Arguably, one of the most important issues – what should the robot do after something has done wrong?

- e.g., part has slipped, environment has significantly changed w.r.t. models



# Also Need to Think of Interactions between Constraints



Good grasp vs. Feasible Gross Motion Plans

Stable grasp vs. Collisions

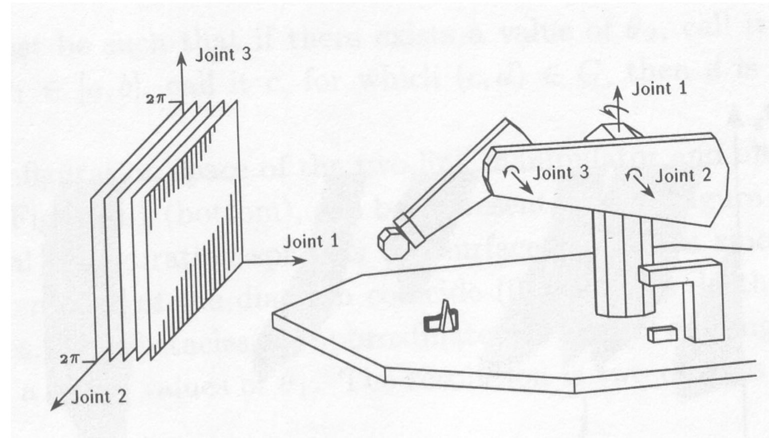
# Many High-Level Concerns

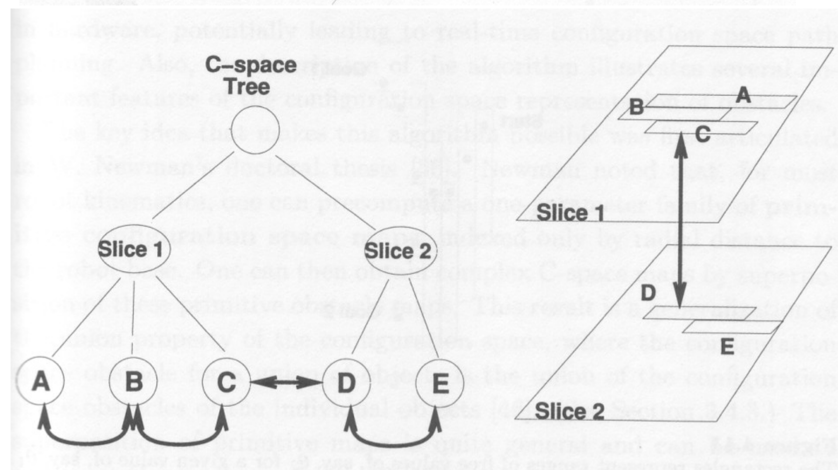
Broadly speaking, even with this simple setup, we can identify patterns of high-level concerns:

- Gross motion planning
- Grasp/Regrasp planning
- Multi-arm coordination

# Gross Motion in Handey

- The c-space is represented in terms of slices
- The free-space is then represented using a tree data structure that utilizes the joint information in the slices
- In those days, these computations were implemented in parallell on a Connection Machine

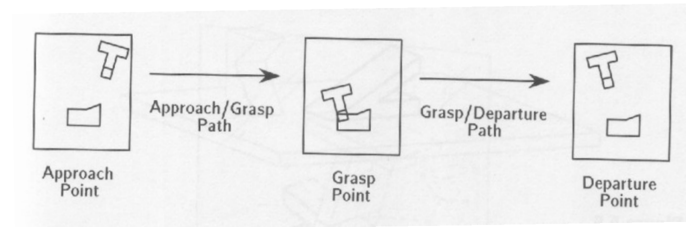


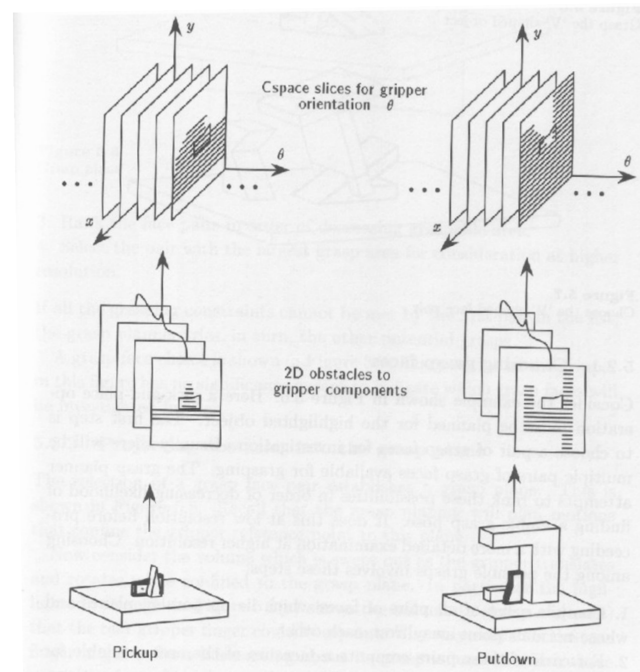



# Grasp Planning in Handey

Broadly, involves the following calculations:

- Choose faces to grasp
- Project obstacles/constraints onto grasp plane
- Characterize grasp and decide on approach/departure
- Search c-space for grasp pose, approach pose, departure pose, approach path, departure path



# Multi-Robot Coordination in Handey

- Concept of a temporal coordination diagram
- A monotonically increasing line corresponds to sequence of poses
- Collisions can be represented as well and hence constraints can be specified