First-Order Logic

Exercise 1:
List all the subformulas (i.e., subexpressions of type Term(bool)) of the following formula. For each subformula \( \phi \), give the set \( \text{free}(\phi) \) of free variables in \( \phi \). You should assume that \( a \) is a constant rather than a variable.

\[
((Q x) \lor \exists y. (P z x) \land (Q a)) \lor \forall z. (R x z x)
\]

Exercise 2:
Using the nltk.sem module, construct a model that jointly satisfies the following sentences. If you can’t construct a model that jointly satisfies all of them, describe why not. Translate the sentences into a form that can be parsed by LogicParser and use the evaluate() function to show the semantic value that each sentence receives in your model. Give a print-out of this evaluation.

\[
(2) \quad \forall x. \exists y. (\text{neighbour } y x)
\]

\[
(3) \quad \forall x. (\neg (\text{neighbour } x x))
\]

\[
(4) \quad \forall x. \forall y. \forall z. ((\text{neighbour } y x) \land (\text{neighbour } z y)) \rightarrow (\text{neighbour } z x)
\]

Exercise 3:
This exercise is meant to remind you that the names for predicates are meaningless, in the sense that the predicates receive their interpretation solely from the valuation function of the model.

Devise a model for each of the following formulas, and then think about real-world concepts that could be ‘translated’ with the predicates \( P \), \( Q_1 \) and \( Q_2 \). That is, think about natural language
sentences that fill in the predicates and intuitively make the formula on the right side of the ‘→’ true when that on the left is true.

(5)  a.  ∀x ∀y.((P y x) → (P x y))
    b.  ∀x ∀y.((Q_1 y x) → (Q_2 x y))

**Exercise 4:**
Consider the following description:

There are exactly four blocks. Two of the blocks are cubical, and two are pyramid shaped. The cubical blocks are the same size, and both are red. The larger of the two pyramids is green, the smaller is yellow. Three of the blocks are sitting directly on the table, while the small pyramid is sitting on a cube.

(a) Devise a suitable vocabulary of non-logical constants and represent the described situation as a first order model.

(b-i) Describe this situation in a first order language using the vocabulary from (a). That is, give a set of first order formulas that is true only in models like the one you gave for (a). (Note that this doesn’t necessarily mean that you have to ‘translate’ every sentence; all you’re asked to do is to describe the situation that is described in English above.)

To get you started, here is how you can say “there are exactly four blocks.”

(6)  ∃x_0.∃x_1.∃x_2.∃x_3.((block x_0) ∧ ... ∧ (block x_3) ∧
    (x_0 ≠ x_1) ∧ ... ∧ (x_2 ≠ x_3) ∧
    ∀y.((block y) → ((y = x_0) ∨ ... ∨ (y = x_3)))

(b-ii) Can you think of models for this set of formulae which differ from the one you gave for (a) and still satisfy the set?

**Semantic Composition**

**Exercise 5:**
The following command will load a simple-minded feature-based grammar from the NLTK distribution:

```python
grammar = nltk.data.load('grammars/sem2.fcfg')
```

This grammar contains a grammar rule for transitive verb phrases and a lexical entry for the verb *chases*, as follows:


**TV**[num=sg,sem=<\X y. (X \x. (chase x y)),tns=pres]> → ‘chases’

(Note that X represents a variable of the same type as quantifier phrases; i.e., ((IND → BOOL) → BOOL).)

Extend the grammar with a rule for ditransitive verbs and a lexical entry for *show* which will capture sentences such as:
Suzie shows John a dog.

In addition, build a model which contains a valuation for show, and demonstrate that it satisfies semantic representations of sentences containing this verb.

Exercise 6:
Transcribe the following sentence into FOL:

(8) Every person needs a doctor.

Build a model that only satisfies one reading. Devise another model that satisfies both readings. What does this tell you about the semantic relationship between the two readings? (You can choose whether or not to implement the model in NLTK.)

Tableaux Proof Methods

Exercise 7:
The connective nor is defined by the following truth tables:

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>nor q</th>
</tr>
</thead>
<tbody>
<tr>
<td>TRUE</td>
<td>TRUE</td>
<td>FALSE</td>
</tr>
<tr>
<td>TRUE</td>
<td>FALSE</td>
<td>FALSE</td>
</tr>
<tr>
<td>FALSE</td>
<td>TRUE</td>
<td>FALSE</td>
</tr>
<tr>
<td>FALSE</td>
<td>FALSE</td>
<td>TRUE</td>
</tr>
</tbody>
</table>

(a) Give the tableaux expansion rules for this connective.

(b) Show that \( \neg(p \lor q) \leftrightarrow (p \text{ nor } q) \) is provable.