RL 5: Markovian decision processes

Michael Herrmann

University of Edinburgh, School of Informatics

28/01/2014

Syllabus

- Reinforcement Learning
- Multi-Armed Bandits
- $\mathcal{Q}$-learning
- Markov Decision Processes
- Dynamic programming
- Monte-Carlo methods
- Back to RL
- POMDPs
- Continuous problems

Markov Chains

The stochastic process is said to have a Markovian property if

$$P(X_{t+1}=j|X_t=i, X_{t-1}=k_{t-1}, \ldots, X_1=k_1, X_0=k_0)=P(X_{t+1}=j|X_t=i)$$

for $t=0,1,\ldots$ and every sequence i, j, $k_0, \ldots, k_{t-1}$

These are *stationary* if time invariant. The we can write

$$p_{ij} = P\left(X_{t+1}=j|X_t=i\right) = P\left(X_1=j|X_0=i\right)$$

$$\mathbf{P}^{(n)} = \begin{pmatrix} p_{00}^{(n)} & \cdots & p_{0M}^{(n)} \\ \vdots & \ddots & \vdots \\ p_{M0}^{(n)} & \cdots & p_{MM}^{(n)} \end{pmatrix}$$

Initial probabilities $\pi_i^0 = P\left\{X_0=i\right\}$ for all i

Markov Chains: First Passage Times

- Number of steps from i to j for the first time is FPT
  - First Passage Times are random variables $\Rightarrow$ mean FPT etc.
  - If $i = j$, this is the *recurrence time*
- n-step recursive relationship for first passage probability

$$
\begin{aligned}
f_{ij}^{(1)} &= p_{ij}^{(1)} = p_{ij} \\
f_{ij}^{(2)} &= p_{ij}^{(2)} - f_{ij}^{(1)} p_{jj} \\
&\vdots \\
f_{ij}^{(n)} &= p_{ij}^{(n)} - f_{ij}^{(1)} p_{jj}^{(n-1)} - f_{ij}^{(2)} p_{jj}^{(n-2)} - \cdots - f_{ij}^{(n-1)} p_{jj}
\end{aligned}
$$

- For fixed i and j, $f_{ij}^{(n)} \geq 0$ and it holds that $\sum_{n=1}^{\infty} f_{ij}^{(n)} \leq 1$
- $\sum_{n=1}^{\infty} f_{ii}^{(n)} = 1$ implies a *recurrent* state; *absorbing* if $f_{ii}^{(1)} = 1$
- mean FPT: $\sum_{n=1}^{\infty} n\, f_{ij}^{(n)}$
- Positive recurrence: State is recurrent and has a mFPT$< \infty$

Markov Chains: Classification of States

- State j is *accessible* from i if $p_{ij}^{(n)} > 0$ (for some n)
- If state j is accessible from i and vice versa, the two states are said to *communicate*
- As a result of communication, one may partition the general Markov chain into states in disjoint classes
  - MC is *irreducible* if there is only one class
- If the MC can only visit the state at integer multiples of t, we call it *periodic*
- Positive recurrent states that are aperiodic are called *ergodic* states

Markov Chains: Long-Run Properties

Inventory example: Interestingly, probability of being in state j after, e.g., 8 weeks appears independent of *initial* level π^0 of inventory.

For an irreducible ergodic Markov chain, one has limiting probability

$$\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j^*$$

i.e. the limit for each element p_{ij} does not depend on i.

$$\pi_j^* = \sum_{i=1}^{M} \pi_i^* p_{ij} \ \forall j = 1, \ldots, M$$

$\pi^* = (\pi_1^*, \ldots, \pi_M^*)$ is an eigenvector of the matrix $P = (p_{ij})$.

Perron-Frobenius theorem: Matrices with positive entries have a unique largest eigenvalue. For a probability matrix this EV is 1.

Reciprocal of π_j^* gives the recurrence time m_{jj}

Markov Chains: Expected Average Cost

Sometimes aperiodic chain is a strong assumption. If we relax it, the limiting probability needs a slightly different definition:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} p_{ij}^{(n)} = \pi_j^*$$

Suppose you incur a (time-independent) cost $C(X_t)$, using above you can derive the long-run expected average over unit time as

$$\lim_{n\to\infty} \left\{ E\left[\frac{1}{n} \sum_{k=1}^{n} C\left(x_t\right) \right] \right\} = \sum_{j=1}^{M} C_j \pi_j^*$$

Can be more elaborate in general, depending on cost function

Markov Decision Model

- Consider the following application: machine maintenance
- A factory has a machine that deteriorates rapidly in quality and output and is inspected periodically, e.g., daily
- Inspection declares the machine to be in four possible states:
  - 0: Good as new
  - 1: Operable, minor deterioration
  - 2: Operable, major deterioration
  - 3: Inoperable
- Let X_t denote this observed state
  - evolves according to some "law of motion", so it is a stochastic process
  - Furthermore, assume it is a finite state Markov chain

Markov Decision Model

- Transition matrix is based on the following:

states	0	1	2	3
0	0	7/8	1/16	1/16
1	0	3/4	1/8	1/8
2	0	0	1/2	1/2
3	0	0	0	1

- Once the machine goes inoperable, it stays there until repairs
  - If no repairs, eventually, it reaches this state which is absorbing!
- Repair is an *action* –- a very simple maintenance *policy*
  - e.g., machine from state 3 to state 0

Markov Decision Model

- There are costs as system evolves:
  - State 0: cost 0
  - State 1: cost 1000
  - State 2: cost 3000
- Replacement cost, taking state 3 to 0, is 4000 (and lost production of 2000), so cost = 6000
- The modified transition probabilities are:

states	0	1	2	3
0	0	7/8	1/16	1/16
1	0	3/4	1/8	1/8
2	0	0	1/2	1/2
3	1	0	0	0

Markov Decision Model

- What is the average cost of this maintenance *policy*?
- Compute the steady state probabilities:

$$\pi_0^* = \frac{2}{13},\ \pi_1^* = \frac{7}{13},\ \pi_2^* = \frac{2}{13},\ \pi_3^* = \frac{2}{13}$$

- (Long run) expected average cost per day

$$0\pi_0^* + 1000\pi_1^* + 3000\pi_2^* + 6000\pi_3^* = \frac{25000}{13} = 1923$$

Markov Decision Model

- Consider a slightly more elaborate policy:
  - Repair when inoperable or replace when needing major repairs
- Permit one more thing: overhaul
  - Go back to minor repairs state (1) for the next time step
  - Not possible if truly inoperable, but can go from major to minor
- Transition matrix now changes a little bit
- Key point about the system behaviour. It evolves according to
  - "Laws of motion"
  - Sequence of decisions made (actions from {1:none, 2:overhaul, 3:replace})
- Stochastic process is now defined in terms of X_t and Δ_t
  - Policy R is a rule for making decisions
  - Could use all history, although popular choice is (current) state-based

Markov Decision Model

- There is a space of potential policies, e.g., (1: none,2:overhaul,3: replace)

Policies	$d_0(R)$	$d_1(R)$	$d_2(R)$	$d_3(R)$
R_a	1	1	1	3
R_b	1	1	2	3
R_c	1	1	3	3
R_d	1	3	3	3

- Each policy defines a transition matrix, e.g., for R_b

states	0	1	2	3
0	0	7/8	1/16	1/16
1	0	3/4	1/8	1/8
2	0	1	0	0
3	1	0	0	0

- Which policy is best? Need costs....

Markov Decision Model

- C_{ik} = expected cost incurred during next transition if system is in state i and decision k is made

state|decision	1	2	3
0	0	4	6
1	1	4	6
2	3	4	6
3	∞	∞	6

- The long run average expected cost for each policy may be computed using

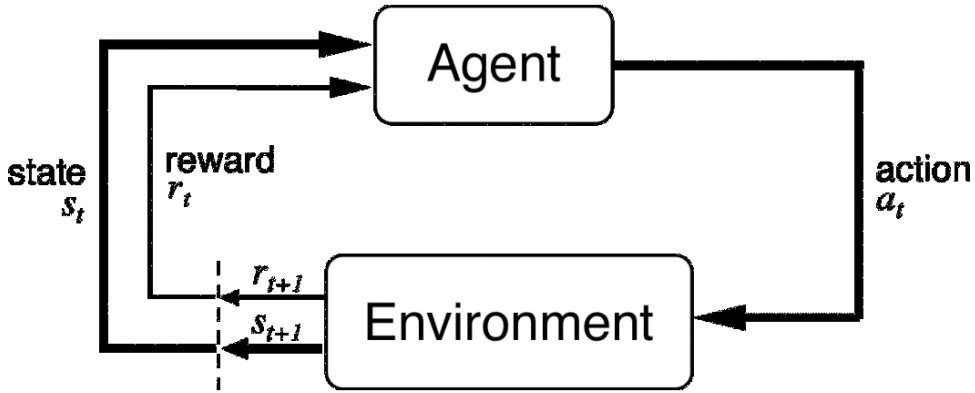
$$E(C) = \sum_{i=0}^{M} C_{ik} \pi_i^*$$

- R_b is best (homework: check!)

Extensions

- Heuristic Search
  - Dynamic Programming: AO*, LAO*, RTDP, ...
- Factored MDPs
  - add planning graph style heuristics
  - use goal regression to generalize better
- Hierarchical MDPs
  - hierarchy of sub-tasks and/or actions to scale better
- Reinforcement Learning
- Partially Observable Markov Decision Processes
  - noisy sensors, partially observable environment, popular in robotics

adapted from Mausam: Markov decision problems, Ch. 17



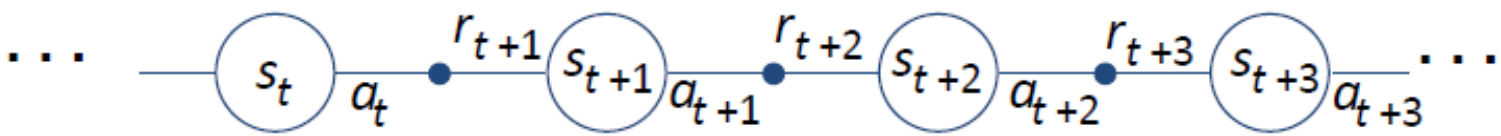
Agent and environment interact at discrete time steps: $t = 0, 1, 2,$

Agent observes state at step t: $s_t \in \mathcal{S}$

Produces action at step t : $a_t \in \mathcal{A}(s_t)$

gets resulting reward: $r_{t+1} \in \mathbb{R}$

and resulting next state s_{t+1}



The Agent Learns a Policy

- Policy at step t, π_t:
  - a mapping from states to action probabilities $\pi_t(s, a)$: probability that $a_t = a$ when $s_t = s$
- Reinforcement learning methods specify how the agent changes its policy as a result of experience.
- Roughly, the agent's goal is to get as much reward as it can over the long run

On the Degree of Abstraction

- Actions can be low level (e.g., voltages to motors), or high level (e.g., accept a job offer), "mental" (e.g., shift in focus of attention), etc.
- States can be low-level "sensations", or they can be abstract, symbolic, based on memory, or subjective (e.g., the state of being "surprised" or "lost").
- An RL agent is not like a *whole* animal or robot.
- Reward computation is in the agent's environment because the agent cannot change it arbitrarily.
- The environment is not necessarily unknown to the agent, only incompletely controllable

Goals and Rewards

- Is a scalar reward signal an adequate notion of a goal? — maybe not, but it is surprisingly flexible
- A goal should specify what we want to achieve, not how we want to achieve it
- A goal must be outside the agent's direct control — thus outside the agent
- The agent must be able to measure success:
  - explicitly;
  - frequently during its lifespan

The reward hypothesis (Sutton)

- That all of what we mean by goals and purposes can be well thought of as the maximisation of the cumulative sum of a received scalar signal (reward)
- A sort of null hypothesis, time scales, stopping criteria needed
- Probably ultimately wrong, but so simple we have to disprove it before considering anything more complicated
- MD problems are solved once optimal values are known

Conclusion

- A fixed policy in a MDP transforms a Markov Chain into a Markov Chain with a (generally) different transition matrix
- $\mathcal{Q}$-learning is not the best example for MDPs
- On-policy reinforcement learning algorithms are often based on MDPs in a strict sense

Where now?

- If the MDP is known, solve the EV-problem, calculate cost for the eigenvector
- Dijkstra's algorithm: visit all states, keep track of distance to starting state
- Assumptions were
  - MDP with known transition probabilities (deterministic for Dijkstra)
  - (random) immediate cost/reward (lengths for Dijkstra)
  - global information is available

Optimality

Bellman's Principle of Optimality: An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision. (1957)

Acknowledgements

Many slides are adapted from web resources associated with Sutton and Barto's Reinforcement Learning book

. . . before being used by Dr. Subramanian Ramamoorthy in this course in the last three years.