Randomness and Computation 2018/19
Week 8 tutorial sheet (12-1pm, Tues 12th, Wed 13th March)

1. We are given an undirected graph $G = (V, E)$ where each $v \in V$ is associated with a set of $8r$ colours $S(v)$, for some $r \geq 1$.

   The $S(v)$ sets may overlap or in some cases be identical, or anything in between; however we have the guarantee that for every $v \in V$ and every $k \in S(v)$, there are at most r neighbours $u \in \mathsf{Nbd}(v)$ that also have $k \in S(u)$.

   Prove that there is a proper (vertex) colouring which assigns a colour from $S(v)$ to each $v \in V$ such that for every $e = (u, v) \in E$, u and v get different colours.

   (in solving this, it may be helpful to define a collection of events $\{A_{u,v,k}\}$, with $A_{u,v,k}$ representing the event that both u and v get the colour k)

2. (Exercise 6.14) Consider the $G_{n,p}$ model for $p = p(n) = \frac{c\ln(n)}{n}$. Use the second moment method (or the conditional expectation inequality of Thm 6.10 from the book) to show that if $c < 1$ then for any constant $\epsilon > 0$ and for n sufficiently large, the graph has isolated vertices (vertices with no neighbours) with probability $\geq (1 - \epsilon)$.

3. Consider the two-state Markov chain with the following transition matrix, assuming the states are named 0 and 1:
$$\mathbf{P} = \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}$$
   Find a simple expression for $P^t_{0,1}$ for general t.

4. In this question we consider a *Markov chain* on the set of *contingency tables*. A collection of *contingency tables* is defined in terms of two lists $r = (r_1, \ldots, r_m)$ and $c = (c_1, \ldots, c_n)$ of positive integer values. These lists are considered to be the *row sums* (the r_is) and *column sums* (the c_js) of hypothetical $m \times n$ tables of non-negative integers. For the given lists r, c, the *set of contingency tables* $\Sigma_{r,c}$ is defined as the set of all $X \in {\mathbb{N}_0}^{mn}$ that satisfy the following conditions:
$$X_{i,j} \geq 0 \quad \text{for all } i \in [m], j \in [n] \tag{1}$$
$$\sum_{j=1}^{n} X_{i,j} = r_i \quad \text{for all } i \in [m] \tag{2}$$
$$\sum_{i=1}^{m} X_{i,j} = c_i \quad \text{for all } j \in [n] \tag{3}$$

   In diagrammatic form, we are interested in all the $m \times n$ tables $X \in {\mathbb{N}_0}^{mn}$ that have the given row rums $r_1, \ldots, r_m$ and given column sums $c_1, \ldots, c_n$:

$X_{1,1}$	$X_{1,2}$	$X_{1,3}$	$\dots$	$X_{1,n}$	r_1
$X_{2,1}$	$X_{2,2}$	$X_{2,3}$	$\dots$	$X_{2,n}$	r_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$X_{m,1}$	$X_{m,2}$	$X_{m,3}$	$\dots$	$X_{m,n}$	r_m
c_1	c_2	c_3	$\dots$	c_n	

Note that our given row and column sums *must* satisfy $\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} c_j$, otherwise we have $\Sigma_{r,c} = \emptyset$ (no feasible solutions).

Above is the description of the set of contingency tables $\Sigma_{r,c}$. We now define our Markov chain on the elements of $\Sigma_{r,c}$. We name the chain M, and for every two contingency tables $X, Y \in \Sigma_{r,c}$, there will be some probability $M[X, Y]$ (maybe 0) that we move from table X to table Y in a single transition.

The transitions (single-step moves) of the Markov chain are a result of the following process: we choose two rows $i, i', i \neq i'$ from $[m]$ independently at random, and two columns $j, j', j \neq j'$ from $[n]$ independently at random. This random draw identifies a "mini-table" of dimensions 2×2 (note i, i' don't have to be adjacent, and neither do j, j'):

$X_{1,1}$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$X_{1,n}$	r_1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\dots$	$\dots$	$\mathbf{X}_{i,j}$	$\dots$	$\mathbf{X}_{i,j'}$	$\dots$	$\dots$	r_i
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\dots$	$\dots$	$\mathbf{X}_{i',j}$	$\dots$	$\mathbf{X}_{i',j'}$	$\dots$	$\dots$	$r_{i'}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$X_{m,1}$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$X_{m,n}$	r_m
c_1	$\dots$	c_j	$\dots$	$c_{j'}$	$\dots$	c_n	

This 2×2 "mini-table" can be visualised independently of the remainder of that table. It will have some individual small row and column sums (depending on the current X, of course):

$X_{i,j}$	$X_{i,j'}$	a_1
$X_{i',j}$	$X_{i',j'}$	a_2
b_1	b_2	

Claim: Focusing on the 2×2 table mapped out by i, i' and j, j', the set of all 2×2 tables with row sums a_1, a_2, and column sums b_1, b_2 is

$$\Sigma_{a,b} = \left\{ \begin{bmatrix} i & (a_1 - i) \\ (b_1 - i) & i + (b_2 - a_1) \end{bmatrix} : \max\{0, a_1 - b_2\} \leq i \leq \min\{a_1, b_1\} \right\}.$$

Transitions: Conditional on having already chosen i, i', j, j' as our "rows and columns", we replace the 2×2 "mini-table" by a uniform random element of $\Sigma_{a,b}$.

The Markov chain M on $\Sigma_{r,c}$: We will have $\mathsf{M}[X, Y] > 0$ for any X, Y such that the non-0 entries of $X - Y$ (the differing entries of X versus Y) lie in some 2×2 sub-matrix. Note this includes the possibility of $X = Y$. Note that for any such pair X, Y that satisfies this condition, the probability $\mathsf{M}[X, Y]$ will be $\frac{2}{m(m-1)} \frac{2}{n(n-1)}$, multiplied by $\frac{1}{|\Sigma_{a,b}|}$ for the specific a_1, a_2, b_1, b_2 of that 2×2 subtable.

If the non-zero values of $X - Y$ *don't* fit into a 2×2 mini-table, then $\mathsf{M}[X, Y] = 0$.

(a) "warm up" for Coursework 2: Consider an example of contingency tables where we have $r = (2, 2, 4), c = (2, 3, 3)$. Suppose that we take the following *state* as our starting state X:

$$X = \begin{array}{ccc||c} \hline 2 & 0 & 0 & 2 \\ \hline 0 & 2 & 0 & 2 \\ \hline 0 & 1 & 3 & 4 \\ \hline\hline 2 & 3 & 3 & \end{array}$$

Work out the subset of contingency tables which can be reached from X in *one transition* of the Markov chain. Also work out the probability of each such transition.

Mary Cryan, 7th March