Randomness and Computation 2018/19
Week 4 tutorial sheet

1. Suppose our goal is to sample the value of a fair coin ("heads" having probability $1/2$, and "tails" having probability $1/2$) but unfortunately we only have a faulty coin, which returns "heads" with some *unknown* probability p, and $1 - p$ otherwise.

   Suppose we do not know the value of p, or even whether $p > 1/2$ or $p < 1/2$. We do assume $p \in (0, 1)$ (neither 0 nor 1). Design a simple algorithm which uses a *number* of coin flips to return a value ("heads" or "tails") in such a way that "heads" and "tails" are each returned with exact probability $1/2$.

   (a) Describe the algorithm.

   (b) Argue/prove that "heads" and "tails" are equally likely to be returned.

   (c) Prove that the *expected* number of coin-flips that will be used by your algorithm is $p^{-1}(1-p)^{-1}$.

2. Consider the following balls-and-bin game. We start with one black ball and one white ball in a bin. We repeatedly do the following: choose one ball from the bin uniformly at random, and then put the ball back in the bin with another ball of the same colour. We repeat until there are n balls in the bin. Show that the number of white balls is equally likely to be any number between 1 and $n - 1$.

3. As we saw in lecture 3, a single "run" of Karger's Min Cut Algorithm has probability at least $\frac{2}{n(n-1)}$ of generating a minimum-sized cut of the given input graph $G = (V, E)$. We also saw that it was possible to run the process a number of times (taking the minimum of all final cuts) to improve the probability of getting an minimum (and that $\Theta(n^2)$ such "runs" would guarantee that we would have a min cut with probability $1 - \frac{1}{n}$).

   We now focus on possibly "re-using" some of the early random draws of Karger's algorithm, and whether that can help us get higher probability of success for the same number of overall random draws. This is Ex 1.25 from Mitzenmacher and Upfal.

   (a) Consider doing just two "runs" of Karger's min cut. How many edge-contractions (equivalent to random draws of edges) will be performed over the two runs?

   (b) Now consider a variation where we start with the original G, and carry out $n - k$ edge contractions to reduce the graph down to G' with k "supervertices", for some given k. Then we store G', and go on to carry out ℓ independent (continuation) "runs" of the Karger algorithm on G', for a given ℓ. This gives rise to ℓ different cuts, though they are not independently generated.

   Determine the number of edge contractions (random draws of edges) done, parametrized by k and ℓ.

   (c) Bound the probability of finding a min cut among the ℓ (dependent) cuts returned by the algorithm in (b). Again, your result will depend on the values of k, ℓ.

(d) Find optimal values for k, ℓ that will optimize the probability of finding a min cut, conditional on the number of edge-contractions of (b) being approximately equal to those for two "runs" of Karger's algorithm (calculated in (a)).

4. Let Y be a nonnegative integer-valued random variable with (strictly) positive expectation. Prove that

$$\frac{(\mathrm{E}[Y])^2}{\mathrm{E}[Y^2]} \leq \Pr[Y \neq 0] \leq \mathrm{E}[Y].$$

Mary Cryan, 31st January