1. We are given an undirected graph \( G = (V, E) \) where each \( v \in V \) is associated with a set of \( 8r \) colours \( S(v) \), for some \( r \geq 1 \).

The \( S(v) \) sets may overlap or in some cases be identical, or anything in between; however we have the guarantee that for every \( v \in V \) and every \( k \in S(v) \), there are at most \( r \) neighbours \( u \in \text{Nbd}(v) \) that also have \( k \in S(u) \).

We now show there is a proper (vertex) colouring which assigns a colour from \( S(v) \) to each \( v \in V \) such that for every edge \( (u, v) \in E \), \( u \) and \( v \) get different colours.

Consider the following procedure for colouring the vertices of \( G \): for each vertex \( v \in V \) we independently pick uniformly at random a colour \( c \in S(v) \), and colour \( v \) with \( c \). We will show that there is a strictly positive probability we end up with a proper colouring of \( G \).

For any pair \( u, v \in V \) such that \( (u, v) \in E \), \( A_{u,v,c} \) denotes the event that both \( u \) and \( v \) are coloured with \( c \). Observe that if \( c \notin S(u) \) or \( c \notin S(v) \), then \( \text{Pr}[A_{u,v,c}] = 0 \). On the other hand, if \( c \in S(u) \) and \( c \in S(v) \), then \( \text{Pr}[A_{u,v,c}] = \frac{1}{8r} = \frac{1}{8r^2} \). Since for each vertex \( v \in V \) and each colour \( c \in S(v) \) there are at most \( r \) neighbours \( u \in \text{Nbd}(v) \) such that \( c \in S(u) \), each event \( A_{u,v,c} \) depends on at most \( d(A_{u,v,c}) = 8r^2 + 8r^2 = 16r^2 \) other events \( A_{u',v',c'} \). Therefore, we have \( \text{Pr}[A_{u,v,c}] \cdot d(A_{u,v,c}) \leq \frac{1}{8r^2} \cdot 16r^2 = \frac{1}{4} \). Hence, by Lovasz Local Lemma we have that \( \text{Pr}[\bigcap_{u,v,c} \overline{A_{u,v,c}}] > 0 \). Therefore, there must exist a colouring of \( G \) such that for any \( (u, v) \in E \), \( u \) and \( v \) have different colours.

2. Let \( I_n \) be the number of isolated vertices in \( G_{n,p} \). We can write \( \text{E}[I_n] = n(1 - p)^{n-1} \). If we have \( p = p(n) = \frac{c \log(n)}{n} \) then we have

\[
\text{E}[I_n] = n \left( 1 - \frac{c \log(n)}{n} \right)^{n-1}
\]

which is approximately \( n e^{-c \log(n)} \), and this itself will be \( n \cdot n^{-c} \). If \( c < 1 \), then this quantity is \( n^{1-c} \) which tends to \( \infty \) as \( n \) grows.

We then need to consider the second moment, and we are lucky that for isolated vertices, the events for two different vertices \( (u, v) \) are almost independent, only depending on the status of the edge between \( u \) and \( v \). We will have

\[
\text{E}[I_n^2] = \sum_{u \in V} \sum_{v \in V} \text{Pr}[I_u I_v] \cdot 1 = \sum_{u \in V} \text{Pr}[I_u] + 2 \sum_{u, v \in V, u \neq v} \text{Pr}[I_u I_v].
\]

We know \( \text{Pr}[I_u] = (1 - p)^{n-1} \) for each \( u \). Also, for each \( u, v, u \neq v \), we know that both vertices are independent iff and only if \( 2(n - 2) + 1 \) edges are omitted (the 1 is for the shared edge \( (u, v) \)), so we have

\[
\text{E}[I_n^2] = n(1 - p)^{n-1} + n(n - 1)(1 - p)^{2n-3}.
\]
Now we use Theorem 6.7 which tells us that \( \Pr[X = 0] \leq \frac{\text{Var}[X]}{E[X]^2} \) for any non-negative r.v \( X \) with bounded \( E[X] \). So for \( I_n \), we have that

\[
\Pr[I_n = 0] \leq \frac{n(1-p)^{n-1} + n(n-1)(1-p)^{2n-3} - n^2(1-p)^{2(n-1)}}{n^2(1-p)^{2(n-1)}}.
\]

This checks out with direct expansions for \( t = 1, 2, 3 \) by using the matrix.

3. We are asked to determine a simple expression for \( P^t[0, 1] \), for \( P \) defined as

\[
P = \begin{bmatrix}
p & 1-p \\
1-p & p
\end{bmatrix}
\]

Note that this Markov chain codes a 1-step process over \( \{0, 1\} \) where \( p \) is the probability that the state keeps its current value, and \( 1-p \) is the probability it flips. If we take the square of this matrix, the value of \( M[0,1] \) would be the probability of \( 0 \rightarrow 1 \) over two steps, which is \( 2p(1-p) \). Carrying forward this analysis, \( M^t[0,1] \) is the probability of an odd number of flips, taken over \( t \) steps of the process. This is \( \sum_{i=1}^{\lfloor t/2 \rfloor} (1-p)^{2i+1} p^{t-1-2i} \binom{t}{2i} \). Inside the summation, it is basically the sum of the odd terms of the binomial expansion.

We can analyse by looking at \( (a + b)^t - (a - b)^t \), which expands as

\[
\left[ \sum_{i=0}^{t} \binom{t}{i} b^i a^{t-i} \right] - \left[ \sum_{i=0}^{t} (-1)^i \binom{t}{i} b^i a^{t-i} \right].
\]

Then the terms with even-\( i \) cancel out, and the terms with odd-\( i \) double, to give the value

\[
2 \left[ \sum_{i'=0}^{\lfloor t/2 \rfloor} \binom{t}{2i'+1} b^{2i'+1} a^{t-2i'-1} \right].
\]

Now return to our sum of terms, and note it matches the expression just-above for \( a = p, b = 1-p \). We have

\[
\frac{1 - (2p - 1)^t}{2} = \sum_{i=0}^{\lfloor t/2 \rfloor - 1} (1-p)^{2i+1} p^{t-1-2i} \binom{t}{2i+1}.
\]

This checks out with direct expansions for \( t = 1, 2, 3 \) by using the matrix.
4. This is a worked example, to help improve understanding of how the chain operates. Our row and column sums are \( r = (2, 2, 4), c = (2, 3, 3) \) for this example, and we have the following *start state* \( X \):

\[
X = \begin{pmatrix}
2 & 0 & 0 & 2 \\
0 & 2 & 0 & 2 \\
0 & 1 & 3 & 4 \\
0 & 3 & 3 \\
\end{pmatrix}
\]

In considering possible transitions, remember that we may choose any pair of rows, any pair of columns, then act on the induced \( 2 \times 2 \) table (keeping the induced small row/column sums of that \( 2 \times 2 \) the same as they started). For a 3-rowed table there are 3 ways to choose a pair of rows, and for a 3-column table 3 ways to choose a pair of columns. So we have \( 3 \times 3 \) different ways of choosing “2 rows and 2 columns”.

Suppose our two rows were \( \{1, 3\} \) and our two columns were \( \{2, 3\} \). This picks out the following highlighted subtable,

\[
X = \begin{pmatrix}
2 & 0 & 0 & 2 \\
0 & 2 & 0 & 2 \\
0 & 1 & 3 & 4 \\
0 & 3 & 3 \\
\end{pmatrix}
\]

with the induced row sums 0, 4 and the induced column sums 1, 3. In this circumstance, with the empty first row, there is only one possible completion of the \( 2 \times 2 \) subtable, which is the original assignment. So in the case where we choose rows \( \{1, 3\} \) and columns \( \{2, 3\} \), we can only transition to our original \( X \).

Note that if we have chosen rows 1, 2 and columns 2, 3, we would again have the value 0 for row 1 of the induced subtable, and again can’t move to any different state.

The same argument above can be applied to a situation where we choose columns 1, 3, and we have row 2 as one of our rows. Then the induced sum on row 2 is 0, and again this “freezes” the original assignment, so again we are forced to stay at our initial \( X \). Note this covers two different subtables, when we take columns 1, 3, and we *either* choose rows 1, 2 or we choose rows 2, 3.

Examining column 1, we can see that any subtable with rows 2, 3 and which has column 1 in its column set, will freeze the induced column sum for column 1m, and hence freeze the rest of the \( 2 \times 2 \) subtable. This covers two cases of the subtable choice, rows 2, 3 with *either* columns 1, 2 or columns 1, 3.

We have shown that 5 of the 9 \( 2 \times 2 \) subtable choices will freeze the table \( X \) (there was overlap between the scenarios that make column 1 induce 0 and make row 2 induce 0).

So there are 4 subtables left, that have some “freedom”. These are rows \( \{2, 3\} \), cols \( \{2, 3\} \); rows \( \{1, 2\} \), cols \( \{1, 2\} \); rows \( \{1, 3\} \), cols \( \{1, 3\} \) and rows \( \{1, 3\} \), cols \( \{1, 2\} \).
In the first case of rows \( \{2, 3\} \), cols \( \{2, 3\} \), we get a little subtable with induced row sums 2, 4 and column sums 3, 3:

\[
X = \begin{pmatrix}
2 & 0 & 0 & 2 \\
0 & 0 & 2 & 2 \\
0 & 1 & 3 & 4 \\
2 & 3 & 3 & 0 \\
\end{pmatrix}.
\]

There are three values that could be placed in entry \((2,2)\) (any of 0, 1, 2) and each of those can be extended to a \(2 \times 2\) subtable with row sums 2, 4 and columns sums 3, 3. So we have \textit{two} new tables different from \(X\) that can be achieved with this choice of subtable, these are the following:

\[
X = \begin{pmatrix}
2 & 0 & 0 & 2 \\
0 & 0 & 2 & 2 \\
0 & 3 & 1 & 4 \\
2 & 3 & 3 & 0 \\
\end{pmatrix},
\]

\[
X = \begin{pmatrix}
2 & 0 & 0 & 2 \\
0 & 1 & 1 & 2 \\
0 & 2 & 2 & 4 \\
2 & 3 & 3 & 0 \\
\end{pmatrix}.
\]

We can continue on in this manner. If we had chosen rows \( \{1, 2\} \) and columns \( \{1, 2\} \), we would have a \(2 \times 2\) subtable with induced row sums 2, 2 and induced column sums 2, 2, and in this situation there are again three completions of this subtable which preserve the row/column sums (giving position \((1, 1)\) value 0, 1, or 2), two of which are different from \(X\).

If we had chosen rows \( \{1, 3\} \), columns \( \{1, 3\} \), then we would have got a subtable with induced row sums 2, 3 and column sums also 2, 3, we again have three completions of this subtable which preserve the row/column sums (giving position \((1, 1)\) value 0, 1, or 2), two of which are different from \(X\).

Finally, had we chosen rows \( \{1, 3\} \), columns \( \{1, 2\} \) we would have got a subtable with induced row sums 2, 1 and column sums 2, 1 and in this case there are just two completions of this subtable which preserve the row/column sums (giving position \((3, 2)\) value 0, 1), just one of which are different from \(X\).

Note that the probability of \(X \rightarrow X'\) for the first six of these different \(X'\) will be \(\frac{1}{3}\) (the probability of selecting that subtable, times the probability of that particular completion), and the 7th different \(X'\) (when we chose rows \( \{1, 3\} \), columns \( \{1, 2\} \)) will have probability \(\frac{11}{32}\).

The probability that we have the transition \(X \rightarrow X\) (ie, no change) is then \(1 - 6 \cdot \frac{11}{32} - \frac{11}{2}\), which is \(\frac{13}{18}\).

Mary Cryan, updated 18th April