Randomness and Computation
or, “Randomized Algorithms”

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Bounding deviation

We already have …

**Theorem (3.1, Markov’s Inequality)**

Let $X$ be any random variable that takes only non-negative values. Then for any $a > 0$,

\[ \Pr[X \geq a] \leq \frac{\mathbb{E}[X]}{a}. \]

And also …

**Theorem (3.2, Chebyshev’s Inequality)**

For every $a > 0$,

\[ \Pr[|X - \mathbb{E}[X]| \geq a] \leq \frac{\text{Var}[X]}{a^2}. \]

These are *generic*. Chernoff/Hoeffding bounds (specific) give tighter bounds for *sums of independent 0/1 variables* and related distributions.

RC (2017/18) – Lecture 7 – slide 2
Chernoff Bounds from the book

*Poisson trials* - sequence of Bernoulli variables $X_i$ with varying $p_i$s.

**Theorem (4.4)**

Let $X_1, \ldots, X_n$ be independent Poisson trials such that $\Pr[X_i = 1] = p_i$ for all $i \in [n]$. Let $X = \sum_{i=1}^{n} X_i$, and $\mu = \mathbb{E}[X]$. We have the following Chernoff bounds:

1. For any $\delta > 0$,
   \[
   \Pr[X \geq (1 + \delta)\mu] \leq \left( \frac{e^\delta}{(1 + \delta)^{1+\delta}} \right)^\mu;
   \]

2. For any $0 < \delta \leq 1$,
   \[
   \Pr[X \geq (1 + \delta)\mu] \leq e^{-\mu \delta^2/3};
   \]

3. For $R \geq 6\mu$,
   \[
   \Pr[X \geq R] \leq 2^{-R}.
   \]

RC (2017/18) – Lecture 7 – slide 3
Chernoff Bounds from the book

Lemma
For $n$ independent Poisson trials $X_1, \ldots, X_n$ and $X = \sum_{i=1}^{n} X_i$, 
$\mu = E[X]$, 

$$E[e^{tX}] \leq e^{\mu(e^t - 1)}.$$  

Proof.
To prove the result, we will consider $E[e^{tX}]$ for $t > 0$. 
This is $E[e^{t(\sum_{i=1}^{n} X_i)}] = E[\prod_{i=1}^{n} e^{tX_i}]$. The $X_i$ and hence the $e^{tX_i}$ are mutually independent, so by Thm 3.3, $E[e^{tX}] = \prod_{i=1}^{n} E[e^{tX_i}]$. 
Each $e^{tX_i}$ has expectation

$$E[e^{tX_i}] = p_i \cdot e^t + (1 - p_i) \cdot 1$$

$$= 1 + p_i(e^t - 1)$$

$$\leq e^{p_i(e^t - 1)} \quad \text{by } 1 + x \leq e^x \text{ for } x \in \mathbb{R}$$

$$E[e^{tX}] \leq \prod_{i=1}^{n} e^{p_i(e^t - 1)} = e^{\sum_{i=1}^{n} p_i(e^t - 1)} = e^{\mu(e^t - 1)}.$$
Lemma

For $n$ independent Poisson trials $X_1, \ldots, X_n$ and $X = \sum_{i=1}^{n} X_i$, $\mu = E[X]$,\[ E[e^{tX}] \leq e^{\mu(e^t-1)}. \]

Proof.

To prove the result, we will consider $E[e^{tX}]$ for $t > 0$.

This is $E[e^{t(\sum_{i=1}^{n} X_i)}] = E[\prod_{i=1}^{n} e^{tX_i}]$. The $X_i$ and hence the $e^{tX_i}$ are mutually independent, so by Thm 3.3, $E[e^{tX}] = \prod_{i=1}^{n} E[e^{tX_i}]$.

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Chernoff Bounds from the book

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$$E[e^{tX_i}] = p_i \cdot e^t + (1 - p_i) \cdot 1$$
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by $1 + x \leq e^x$ for $x \in \mathbb{R}$

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Lemma
For $n$ independent Poisson trials $X_1, \ldots, X_n$ and $X = \sum_{i=1}^{n} X_i$, 
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$$\mathbb{E}[e^{tX}] \leq e^{\mu(e^t - 1)}.$$ 

Proof.
To prove the result, we will consider $\mathbb{E}[e^{tX}]$ for $t > 0$. 
This is $\mathbb{E}[e^{t(\sum_{i=1}^{n} X_i)}] = \mathbb{E}[\prod_{i=1}^{n} e^{tX_i}]$. The $X_i$ and hence the $e^{tX_i}$ are mutually independent, so by Thm 3.3, $\mathbb{E}[e^{tX}] = \prod_{i=1}^{n} \mathbb{E}[e^{tX_i}]$. 
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\[ \mu = E[X], \]
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**Lemma**

For $n$ independent Poisson trials $X_1, \ldots, X_n$ and $X = \sum_{i=1}^{n} X_i$, 

$\mu = \mathbb{E}[X]$, 

$$
\mathbb{E}[e^{tX}] \leq e^{\mu(e^t-1)}.
$$

**Proof.**

To prove the result, we will consider $\mathbb{E}[e^{tX}]$ for $t > 0$.

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$$

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$$
Chernoff Bounds from the book

Proof of Thm 4.4 (1.)

Interested in events when $X \geq (1 + \delta)\mu$.

Identical to when $e^X \geq e^{(1+\delta)\mu}$, or for any $t > 0$, when $e^{tX} \geq e^{t(1+\delta)\mu}$.

\[ \Pr[X \geq (1 + \delta)\mu] = \Pr[e^{tX} \geq e^{t(1+\delta)\mu}] \]
\[ \leq \frac{E[e^{tX}]}{e^{t(1+\delta)\mu}} \quad \text{by Markov's Inequality} \]
\[ \leq \frac{e^{\mu(e^t - 1)}}{e^{t(1+\delta)\mu}} \quad \text{by Lemma just proved} \]

Now take $t = \ln(1 + \delta)$ (and note this is $> 0$) to see

\[ \Pr[X \geq (1 + \delta)\mu] \leq \frac{e^{\mu(e^{\ln(1+\delta)} - 1)}}{e^{\ln(1+\delta)(1+\delta)\mu}} \]
\[ = \frac{e^{\mu\delta}}{(1 + \delta)(1+\delta)\mu} = \left(\frac{e^{\delta}}{(1 + \delta)(1+\delta)}\right)^\mu. \]
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\Pr[X \geq (1 + \delta)\mu] = \Pr[e^{tX} \geq e^{t(1+\delta)\mu}]
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RC (2017/18) – Lecture 7 – slide 5
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$$Pr[X \geq (1 + \delta)\mu] = Pr[e^{tX} \geq e^{t(1+\delta)\mu}]$$

$$\leq \frac{E[e^{tX}]}{e^{t(1+\delta)\mu}} \quad \text{by Markov’s Inequality}$$

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\Pr[X \geq (1 + \delta)\mu] = \Pr[e^{tX} \geq e^{t(1+\delta)\mu}] \\
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Chernoff Bounds from the book

Proof of Thm 4.4 (2.)

Already have

\[\Pr[X \geq (1 + \delta)\mu] \leq \left(\frac{e^\delta}{(1 + \delta)(1+\delta)}\right)^\mu.\]

The rhs will be \(\leq e^{-\mu\delta^2/3}\) if and only if (taking \(\mu\)-th root, then ln)

\[\delta - (1 + \delta)\ln(1 + \delta) < -\delta^2/3\]

We will show the following \(f\) is always negative for \(\delta \in (0, 1)\)

\[f(\delta) =_{def} \delta - (1 + \delta)\ln(1 + \delta) + \delta^2/3\]

Differentiating,

\[f'(\delta) = 1 - \ln(1 + \delta) - (1 + \delta) \frac{1}{1 + \delta} + \frac{2\delta}{3} = -\ln(1 + \delta) + \frac{2\delta}{3}\]
Chernoff Bounds from the book

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Chernoff Bounds from the book

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Chernoff Bounds from the book

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\[ -\ln(1 + \delta) + \_ \]
Chernoff Bounds from the book

Proof of Thm 4.4 (2.) cont’d.

\[ f'(\delta) = -\ln(1 + \delta) + \frac{2\delta}{3}. \]

Differentiating again

\[ f''(\delta) = -\frac{1}{1+\delta} + \frac{2}{3} = -\frac{1}{1+\delta} + \frac{2}{3} \]

Note

\[ f''(\delta) \begin{cases} < 0 & \text{for } 0 < \delta < 1/2 \\ 0 & \delta = 1/2 \\ > 0 & \delta > 1/2 \end{cases} \]

Also \( f'(0) = 0, f'(1) < 0 \) (check \( \delta = 1 \) in top equation), and by \( f' \) decreasing first, then increasing from \( 1/2 \) \( f'(\delta) < 0 \) on \( (0, 1) \). By \( f(0) = 0 \), this implies that \( f(\delta) \leq 0 \) in all of \([0, 1]\). Hence \( \delta - (1 + \delta) \ln(1 + \delta) < -\delta^2/3 \), proving 2.

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Chernoff Bounds from the book

Proof of Thm 4.4 (2.) cont’d.

\[ f'(\delta) = -\ln(1 + \delta) + \frac{2\delta}{3}. \]

Differentiating again

\[ f''(\delta) = -\frac{1}{1 + \delta} + \frac{2}{3} = -\frac{1}{1 + \delta} + \frac{2}{3} \]

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\[ f'(\delta) = -\ln(1 + \delta) + \frac{2\delta}{3}. \]

Differentiating again

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\[
\begin{align*}
    f''(\delta) \left\{ 
    \begin{array}{ll}
        < 0 & \text{for } 0 < \delta < 1/2 \\
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    \end{array}
    \right.
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Also \( f'(0) = 0, f'(1) < 0 \) (check \( \delta = 1 \) in top equation), and by \( f' \) decreasing first, then increasing from \( 1/2 \) \( f'(\delta) < 0 \) on \( (0, 1) \).

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Proof of Thm 4.4 (2.) cont’d.

\[ f'(\delta) = -\ln(1 + \delta) + \frac{2\delta}{3}. \]

Differentiating again

\[ f''(\delta) = -\frac{1}{1 + \delta} + \frac{2}{3} = -\frac{1}{1 + \delta} + \frac{2}{3} \]

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\begin{align*}
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decreasing first, then increasing from \( 1/2 \) \( f'(\delta) < 0 \) on \( (0, 1) \).
By \( f(0) = 0 \), this implies that \( f(\delta) \leq 0 \) in all of \([0, 1]\).
Hence \( \delta - (1 + \delta) \ln(1 + \delta) < -\delta^2/3 \), proving 2.
Theorem (4.5)

Let $X_1, \ldots, X_n$ be independent Poisson trials such that $\Pr[X_i = 1] = p_i$ for all $i \in [n]$. Let $X = \sum_{i=1}^{n} X_i$, and $\mu = \mathbb{E}[X]$. For any $0 < \delta < 1$, we have the following Chernoff bounds:

1. 
   $$\Pr[X \leq (1 - \delta)\mu] \leq \left(\frac{e^{-\delta}}{(1 - \delta)^{1-\delta}}\right)^\mu;$$

2. 
   $$\Pr[X \leq (1 - \delta)\mu] \leq e^{-\mu \delta^2 / 2};$$

- Proof is similar to Thm 4.4.
- Bound of 2. is slightly better than for the $\geq (1 + \delta)\mu$ bound.
- No 3. Why?
Concentration

**Corollary (4.6)**

Let $X_1, \ldots, X_n$ be independent Poisson trials such that $\Pr[X_i = 1] = p_i$ for all $i \in [n]$. Let $X = \sum_{i=1}^{n} X_i$, and $\mu = E[X]$. Then for any $\delta, 0 < \delta < 1$,

$$\Pr[|X - \mu| \geq \delta \mu] \leq 2e^{-\mu \delta^2/3}.$$ 

- For almost all applications, we will want to work with a symmetric version like the Corollary.
- We “threw away” a bit in moving from the $(e^{\pm \delta}(1 \pm \delta)^{1 \pm \delta})^\mu$ versions, but they are tricky to work with.
Analysing a collection of coin flips

Suppose we have \( p_i = 1/2 \) for all \( i \in [n] \).
We have \( \mu = E[X] = \frac{n}{2} \), \( \text{Var}[X] = \frac{n}{4} \).
Consider the probability of being further than \( 5\sqrt{n} \) from \( \mu \).

**Chebyshev** \( \Pr[|X - \mu| \geq 5\sqrt{n}] \leq \frac{\text{Var}[X]}{25n} = \frac{1}{100} \)

**Chernoff** Work out the \( \delta \) - we need \( \mu \delta = 5\sqrt{n} \), so need \( \delta = 5\sqrt{n}/\mu = 10\sqrt{n}/n = \frac{10}{\sqrt{n}} \). Then by Chernoff

\[
\Pr[|X - \mu| \geq 5\sqrt{n}] \leq 2e^{-\mu \delta^2/3} = 2e^{\frac{-10^2 \cdot n}{2 \cdot 3 \cdot \sqrt{n}^2}} = 2e^{-16.6\ldots}.
\]

This is much smaller than the Chebyshev bound (though note it doesn’t depend on \( n \)).

Get much improved bounds because Chernoff uses specialised analysis for sums of independent Bernoulli variables.
The handout for this lecture had extra slides in it at the end. I didn’t finish in class on 6th Feb so have moved those slides to lecture 8.

- Chapter 4 of “Probability and Computing"
- We don’t have time to cover the packet routing analysis of 4.5. But it’s worth reading (but not examinable in the exam).