**Chernoff Bounds from the book**

**Poisson trials - sequence of Bernoulli variables** \(X_i\) with varying \(p_i\)s.

**Theorem (4.4)**

Let \(X_1, \ldots, X_n\) be independent Poisson trials such that \(\Pr[X_i = 1] = p_i\) for all \(i \in [n]\). Let \(X = \sum_{i=1}^n X_i\), and \(\mu = \mathbb{E}[X]\). We have the following Chernoff bounds:

1. For any \(\delta > 0\),

\[
\Pr[X \geq (1 + \delta)\mu] \leq \left( \frac{e^\delta}{(1 + \delta)^{1+\delta}} \right)^\mu.
\]

2. For any \(0 < \delta \leq 1\),

\[
\Pr[X \geq (1 + \delta)\mu] \leq e^{-\mu\delta^2/3};
\]

3. For \(R \geq 6\mu\),

\[
\Pr[X \geq R] \leq 2^{-R}.
\]

**Lemma**

For \(n\) independent Poisson trials \(X_1, \ldots, X_n\) and \(X = \sum_{i=1}^n X_i\),

\[
\mathbb{E}[e^{tX}] \leq e^{\mu(t^2-1)}.
\]

**Proof.**

To prove the result, we will consider \(\mathbb{E}[e^{tX}]\) for \(t > 0\).

This is \(\mathbb{E}[e^{t(\sum_{i=1}^n X_i)}] = \mathbb{E}[\prod_{i=1}^n e^{tX_i}]\).

The \(X_i\) are mutually independent, so by Thm 3.3, \(\mathbb{E}[e^{tX}] = \prod_{i=1}^n \mathbb{E}[e^{tX_i}]\).

Each \(e^{tX_i}\) has expectation

\[
\mathbb{E}[e^{tX_i}] = p_i \cdot e^t + (1 - p_i) \cdot 1 \leq e^{t(f^i - 1)} \quad \text{by} \quad 1 + x \leq e^x \quad \text{for} \quad x \in \mathbb{R}
\]

\[
\mathbb{E}[e^{tX}] \leq \prod_{i=1}^n e^{p_i(t^2-1)} = e^{\sum_{i=1}^n p_i(t^2-1)} = e^{\mu(t^2-1)}.
\]
Chernoff Bounds from the book

Proof of Thm 4.4 (1.)
Interested in events when $X \geq (1 + \delta)\mu$.
Identical to when $e^X \geq e^{(1 + \delta)\mu}$, or for any $t > 0$, when $e^{tX} \geq e^{(1 + \delta)\mu}$.

\[
\Pr[X \geq (1 + \delta)\mu] = \frac{\Pr[e^{tX} \geq e^{(1 + \delta)\mu}]}{e^{(1 + \delta)\mu}} \leq \frac{\mathbb{E}[e^{tX}]}{e^{(1 + \delta)\mu}} \quad \text{by Markov's Inequality}
\]

\[
\leq \frac{e^\mu}{e^{(1 + \delta)\mu}} = \left(\frac{e^\delta}{1 + \delta}\right)^\mu.
\]

Now take $t = \ln(1 + \delta)$ (and note this is $> 0$) to see

\[
\Pr[X \geq (1 + \delta)\mu] \leq \frac{e^\mu}{(1 + \delta)^{(1 + \delta)\mu}} = \left(\frac{e^\delta}{1 + \delta}\right)^\mu.
\]

Chernoff Bounds from the book

Proof of Thm 4.4 (2.) cont'd.
\[
f'(\delta) = -\ln(1 + \delta) + \frac{2\delta}{3}.
\]

Differentiating again
\[
f''(\delta) = -\frac{1}{1 + \delta} + \frac{2}{3} = -\frac{1}{1 + \delta} + \frac{2}{3}.
\]

Note
\[
f''(\delta) \begin{cases} < 0 & \text{for } 0 < \delta < 1/2 \\ 0 & \delta = 1/2 \\ > 0 & \delta > 1/2 \end{cases}
\]

Also $f'(0) = 0, f'(1) < 0$ (check $\delta = 1$ in top equation), and by $f'$ decreasing first, then increasing from $1/2$ $f''(\delta) < 0$ on $(0, 1)$. By $f(0) = 0$, this implies that $f(\delta) \leq 0$ in all of $[0, 1]$. Hence $\delta - (1 + \delta)\ln(1 + \delta) < -\delta^2/3$, proving 2. \(\square\)

Chernoff Bounds from the book (other direction)

Theorem (4.5)
Let $X_1, \ldots, X_n$ be independent Poisson trials such that $\Pr[X_i = 1] = p_i$ for all $i \in [n]$. Let $X = \sum_{i=1}^n X_i$, and $\mu = \mathbb{E}[X]$. For any $0 < \delta < 1$, we have the following Chernoff bounds:

1. \[
\Pr[X \leq (1 - \delta)\mu] \leq \left(\frac{e^{-s}}{(1 - \delta)^{1-\delta}}\right)^\mu;
\]

2. \[
\Pr[X \leq (1 - \delta)\mu] \leq e^{-\mu \delta^2/2};
\]

- Proof is similar to Thm 4.4.
- Bound of 2. is slightly better than for the $\geq (1 + \delta)\mu$ bound.
- No 3. Why?
Concentration

Corollary (4.6)
Let $X_1, \ldots, X_n$ be independent Poisson trials such that $\Pr[X_i = 1] = p_i$ for all $i \in [n]$. Let $X = \sum_{i=1}^{n} X_i$, and $\mu = E[X]$. Then for any $\delta, 0 < \delta < 1$,

$$\Pr[|X - \mu| \geq \delta \mu] \leq 2e^{-\mu \delta^2 / 3}.$$

For almost all applications, we will want to work with a symmetric version like the Corollary.

We “threw away” a bit in moving from the $(e^{\pm \delta} (1 \pm \delta) 1 \pm \delta) \mu$ versions, but they are tricky to work with.

Analysing a collection of coin flips

Suppose we have $p_i = 1/2$ for all $i \in [n]$.
We have $\mu = E[X] = \frac{n}{2}$, $\text{Var}[X] = \frac{n}{4}$.
Consider the probability of being further than $5\sqrt{n}$ from $\mu$.

**Chebyshev**

$$\Pr[|X - \mu| \geq 5\sqrt{n}] \leq \frac{\text{Var}[X]}{\mu^2} = \frac{1}{100}.$$

**Chernoff**

Work out the $\delta$ - we need $\mu \delta = 5\sqrt{n}$, so need

$$\delta = 5\sqrt{n} / \mu = 10\sqrt{n} / n = \frac{10}{\sqrt{n}}.$$ Then by Chernoff

$$\Pr[|X - \mu| \geq 5\sqrt{n}] \leq 2e^{-\mu \delta^2 / 3} = 2e^{-\frac{100n}{3}} = 2e^{-16.6\ldots}.$$ This is much smaller than the Chebyshev bound (though note it doesn’t depend on $n$).

Get much improved bounds because Chernoff uses specialised analysis for sums of independent Bernoulli variables.

References

- Chapter 4 of “Probability and Computing”
- We will continue with Chernoff Bounds on Friday
- We may not have time to cover the packet routing analysis of 4.5. But it’s worth reading (but not examinable in the exam).