Randomness and Computation
or, “Randomized Algorithms”

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Discrete Random Variables

Much of our reasoning in RC is in terms of random variables, especially discrete random variables (when $X$ can take on a finite or countable number of values). I assume standard definitions (and known facts!).

Not all random variables have bounded expectation. Expectation is finite if $\sum_i |i| \Pr[X = i]$ converges as a series; otherwise unbounded. (note that $X$ cannot be unbounded unless it has infinite support).

Theorem (2.1, Linearity of Expectation)

For any finite collection of discrete random variables $X_1, \ldots, X_k$ with finite expectations,

$$E \left[ \sum_{j=1}^{k} X_j \right] = \sum_{j=1}^{k} E[X_j].$$

Theorem 2.1 holds regardless of whether the random variables are independent or not.
Lemma (2.2)

For any discrete random variable $X$, any constant $c$, $E[c \cdot X] = c \cdot E[X]$.

Definition (2.2)

A collection $X_1, \ldots, X_k$ of random variables are said to be \textit{mutually independent} if and only if, for every subset $I \subseteq \{1, \ldots, k\}$, and every tuple of values $a_i, i \in I$, we have

$$
\Pr[\bigcap_{i \in I}(X_i = a_i)] = \prod_{i \in I} \Pr[X_i = a_i].
$$

\textbf{Stronger} than “pairwise independent" - a collection of random variables can be pairwise independent but \textit{not} mutually independent.

Example

Two fair coins, values 1 and 0. $A$ “value of first flip", $B$ “value of second flip", $C$ “absolute difference of two values". Pairwise relationships work out but $\Pr[(A = 1) \cap (B = 1) \cap (C = 1)] = ?$. 
Variance and Second moment

A “partner measure" to expectation (the “first moment”) is variance (or the related measure called the second moment).

Definition
For any discrete random variable $X$, the second moment is defined as $E[X^2]$, ie, $\sum_i i^2 \Pr[X = i]$ ($i$ ranging over the support of $X$). The variance is defined as $E[(X - E[X])^2]$, ie, $\sum_i (i - E[X])^2 \Pr[X = i]$.

Lemma
For any discrete random variable $X$, $E[X^2] \geq E[X]^2$.

Proof.
Define $Y = (X - E[X])^2$, $Y$ is also a discrete random variable. Also $Y$ only takes non-negative values, hence $E[Y] \geq 0$.
By Lemma 2.2, $E[X \cdot E[X]] = E[X]^2$.
Hence $E[Y] = E[X^2] - E[X]^2$, and by $E[Y] \geq 0$, we have $E[X^2] \geq E[X]^2$.  

RC (2017/18) – Lecture 5 – slide 4
Jensen’s Inequality

Definition
A function $f : \mathbb{R} \to \mathbb{R}$ is said to be convex if it is the case that for every $x_1, x_2 \in \mathbb{R}$ and every $\lambda \in [0, 1],$

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2).$$

Lemma (2.3)
For any $f$ which is twice differentiable, $f$ is convex around $x$ if and only if $f''(x) \geq 0.$

Theorem (2.4, Jensen’s Inequality)
If $f$ is a convex function, then

$$E[f(X)] \geq f(E[X]).$$
Jensen’s Inequality

Theorem (2.4, Jensen’s Inequality)

*If* $f$ *is a convex function, then*

$$
E[f(X)] \geq f(E[X]).
$$

**Proof.**

Let $\mu = E[X]$.

Assuming that $f$ is twice differentiable on its domain, then Taylor’s theorem implies there is some value $c \in (\mu, x)$ such that

$$
f(x) = f(\mu) + f'(\mu)(x - \mu) + f''(c)\frac{(x - \mu)^2}{2}.
$$

By convexity of $f$, we know $f''(\cdot) \geq 0$ throughout domain, so $f(x) \geq f(\mu) + f'(\mu)(x - \mu)$ for all $x$. Take $E$, apply Thm 2.1, Lem 2.2,

$$
E[f(X)] \geq E[f(\mu)] + E[f'(\mu)(X - \mu)] = f(E[X]) + f'(E[X]) \cdot (E[X] - E[X])
$$

so the $f'$ term disappears and $E[f(X)] \geq f(E[X])$.  

RC (2017/18) – Lecture 5 – slide 6
Simple distributions

Definition
The *Bernoulli distribution* (biased coin-flip) is the random variable $Y$ such that $Y = 1$ with probability $p$ and $Y = 0$ with probability $1 - p$.

Notice $E[Y] = p$ when $Y$ is Bernoulli.

Definition (2.5)
The *binomial distribution* for $n, p$, written $B(n, p)$, is the random variable $X$ which takes values in $\{0, 1, \ldots, n\}$ with the probabilities

$$\Pr[X = j] = \binom{n}{j} p^j (1 - p)^{n-j}.$$ 

We can prove $E[X] = np$ for $X$ being $B(n, p)$ in *two ways*:

- Directly, using Definition 2.5 and simplifying/summing the series.
- Binomial distribution $B(n, p)$ is the probabilities of getting $j$ flips from $n$ independent trials of a Bernoulli. Then use linearity of expectation.
Conditional Expectation

Definition (2.6)
For two random variables $Y, Z$,

$$E[Y | Z = z] = \sum_y y \cdot Pr[Y = y | Z = z],$$

summation being taken over all $y$ in the support of $Y$.

Lemma (2.5)
For any random variables $X$ and $Y$,

$$E[X] = \sum_y Pr[Y = y] \cdot E[X | Y = y],$$

sum taken over the support of $Y$, and we assume every $E[X | Y = y]$ is bounded.

Proof.
On visualiser.
Conditional Expectation

Observation

For any finite collection of discrete random variables $X_1, \ldots, X_n$ with finite expectations, and for any random variable $X$,

$$
E \left[ \left( \sum_{i=1}^{n} X_i \right) \bigg| Y = y \right] = \sum_{i=1}^{n} E[X_i \big| Y = y].
$$

Definition (2.7)

We will sometimes use the expression $E[Y \big| Z]$, where $Y, Z$ are existing random variables. $E[Y \big| Z]$ itself is a random variable which is a function of $Z$, having the value $E[Y \big| Z = z]$ when applied to $z$. 
Geometric distributions

Imagine we flip a biased coin many times (success with prob. \( p \)), and stop when we see the first success (heads, or alternatively 1). What is the distribution of the number of flips?

**Definition (2.8)**

A geometric random variable \( X \) with parameter \( p \) is given by the following probability distribution on \( \mathbb{N} \):

\[
\Pr[X = j] = (1 - p)^{j-1} p.
\]

Should verify that \( \sum_{j=1}^{\infty} \Pr[X = j] = 1 \) (on visualiser).

Geometric random variables are memoryless (like Markov chains . . .):

**Lemma (2.8)**

For a geometric random variable \( X \) with parameter \( p \), and for any \( j > 0, k \geq 0 \),

\[
\Pr[X = j + k | X > k] = \Pr[X = j].
\]
Geometric distributions

Lemma (2.9)

For any discrete random variable $X$ that only takes non-negative integer values, we have the following:

$$E[X] = \sum_{i=1}^{\infty} \Pr[X \geq i].$$

Proof.
Like this one, will do on visualiser.

Observation

If $X$ is a geometric random variable $X$ with parameter $p$, then for any $i \geq 0$, $\Pr[X \geq i] = (1 - p)^{i-1}$.

Proof.

We have $\Pr[X \geq i] = \sum_{j=i}^{\infty} (1 - p)^{j-1} \cdot p = p \sum_{j=i}^{\infty} (1 - p)^{j-1}$.

Sum $\sum_{j=i}^{\infty} (1 - p)^{j-1}$ as $(1 - p)^{i-1} \frac{1-(1-p)}{1-(1-p)} = (1 - p)^{i-1} p^{-1}$.

Hence $\Pr[X \geq i] = p \cdot (1 - p)^{i-1} p^{-1} = (1 - p)^{i-1}$.  

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Lemma
If $X$ is a geometric random variable with parameter $p$, then $E[X] = p^{-1}$.

Proof.
We just apply Lemma 2.9.
We have $E[X] = \sum_{i=1}^{\infty} \Pr[X \geq i]$.
For a geometric random variable, parameter $p$,

$$E[X] = \sum_{i=1}^{\infty} (1 - p)^{i-1} = \sum_{i=0}^{\infty} (1 - p)^i,$$

which is (using closed form for geometric series again)

$$\frac{1 - (1 - p)^\infty}{1 - (1 - p)} = \frac{1}{p} = p^{-1}.$$
“Coupon collecting" is the activity of buying cereal-packets, each of which will have a coupon inside. There are be $n$ different types of “coupon" (eg cards with a photo of a footballer) and the goal is to collect one copy of each … then stop buying.

How many packets do we (expect to) need to buy?

Assumptions:

- Items are randomly and identically distributed in packets (one card per packet). So when buying a box the probability of any particular card being inside is $1/n$. 
Coupon Collector Analysis

How to analyse the process?

Could evaluate expected number of purchases to get card $j$ (that particular footballer), for any $1 \leq i \leq n$. The “number of steps” $Y_j$ is a geometric random variable with parameter $1/n$. By our Lemma from Tuesday, $E[Y_j] = \frac{1}{(1/n)} = n$ for any $j$.

But this is $n \ Y_j$ variables in total, which are not independent (WHY?). Hard to combine them for a tight estimate. Better to find another angle . . . not focused on any particular card. So . . .

- At any stage of the process (having found some cards already), analyse the “further purchases” to get a card not seen before.
- Let $X_i$ be the number of packets bought (after having $i - 1$ different cards) to get the $i$th new card.
- Let $X$ be the number of packets bought to get all cards.
- Clearly $X = \sum_{i=1}^{n} X_i$. 

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Coupon Collector Analysis

$X_i$ can also be modelled as a geometric random variable; if we own $i - 1$ different cards, and buy one more packet, the (conditional) probability $p_i$ that we get a new card is $p_i = \frac{n - (i-1)}{n} = 1 - \frac{i-1}{n}$.

Linearity of $E[.]$ says $E[X] = \sum_{i=1}^{n} E[X_i]$. 

By Lemma on geometric random variables $E[X_i] = \frac{n}{n-(i-1)}$ for every $i$.

Hence $E[X] = \sum_{i=1}^{n} \frac{n}{n-(i-1)} = \sum_{i=1}^{n} \frac{n}{n} \cdot \frac{1}{i} = n(\sum_{i=1}^{n} \frac{1}{i})$.

$H(n) = \sum_{i=1}^{n} \frac{1}{i}$ is a crude “Riemann sum” to approximate $\int_{x=1}^{n} \frac{1}{x}$.

Can show $\int_{x=1}^{n} \frac{1}{x} < \sum_{i=1}^{n} \frac{1}{i}$ and $\sum_{i=2}^{n} \frac{1}{i} < \int_{x=1}^{n} \frac{1}{x}$ (Fig 2.1 in book).

Hence $\ln(n) < \sum_{i=1}^{n} \frac{1}{i} \leq \ln(n) + 1$.

So the expected time $E[X]$ to collect all cards is $\sim n \ln(n)$.
Is “expected” the same as “typical”?

All we know (for Coupon collecting) is the “average" (weighted over random choices) number of cards.

We don’t know how likely a example “run" of the process is to come close to that value.

Inequalities like Markov’s Inequality, Chebyshev’s Inequality, Chernoff/Hoeffding Bounds help us bound deviation from the mean.

Have a look in Chapter 3 of “Probability and Computing" to remind yourself of the Markov, Chebyshev etc inequalities before Friday.