Randomness and Computation
or, “Randomized Algorithms”

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(Based on slides by M. Cryan)

Karger’s contraction algorithm


Repeatedly, choose an edge uniformly at random (from the not-yet contracted edges) and contract its endpoints.
When there are just two “vertices” left, return that cut.

We will show that this algorithm finds the minimum cut with high probability in time $O(n^2 \log n)$.

Example

The min cut has size 2.

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The algorithm randomly picks one edge out of 14.
We hope to avoid the min cut.
In this case the “bad” thing happens with probability $\frac{2}{14}$.

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merge the endpoints of an edge into one.
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If we contract a cut edge, then we will not find that cut.

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Implementation of the algorithm

Naive implementation of contractions would require complicated data structures to keep track of everything.

An equivalent way of looking at the algorithm is to pick a random permutation of all edges first, and then contracting edges from the first to the last.

Thus, what we need to find is the shortest prefix of the permutation such that they induce two connected components.

Finding connected components takes $O(m)$ time. Thus, by a binary search, this will take time

$$O(m) + O(m/2) + O(m/4) + \cdots = O(m).$$

Denote by $\text{KargerOneTrial}$ one iteration of the above.
Karger’s contraction algorithm - analysis

Let \( k \) be the size of a min cut of \( G \), let \( S \subset V \) be a specific partition where \( C_S \), the set of edges between \( S \) and \( V \setminus S \), is of cardinality \( k \).

We must have \( \deg(v) \geq k \) for every \( v \in V \). (Why?)

The algorithm chooses a sequence of random edges \( e_1, e_2, \ldots \)

Let \( E_j \) be the event that \( e_j \notin C_S \). (“Good” event.)

Calculating \( \Pr[E_j] \), there are \( k \) “cut-edges” (from \( C_S \)), and at least \( k \cdot n/2 \) edges overall. Hence

\[
\Pr[E_j] \geq 1 - \frac{2k}{kn} \geq 1 - \frac{2}{n}.
\]

We next calculate \( \Pr[E_2 \mid E_1] \), the probability that the 2nd edge avoids \( C_S \), conditional that the first edge was outside \( C_S \).
Karger’s contraction algorithm - analysis

\[
\Pr[E_2 \mid E_1]:
\]

- Still have all \( k C_S \) edges (since we assumed \( E_1 \)).
- Graph now has \((n - 1)\) “vertices”, each having degree \( \geq k \) (why?); hence the graph now has at least \( k \cdot (n - 1)/2 \) edges overall.

Hence

\[
\Pr[E_2 \mid E_1] \geq 1 - \frac{2k}{(n - 1)k} = 1 - \frac{2}{n - 1}.
\]

Next we will generalise this bound, namely, for any initial sequence of \( j \) edge-choices satisfying \( \cap_{i=1}^j E_i \), we give a lower bound on

\[
\Pr[E_{j+1} \mid E_1 \cap \ldots \cap E_j].
\]

Karger’s contraction Algorithm - Analysis

For any \( j = 1, \ldots, n - 3 \), we analyse the conditional probability \( \Pr[E_{j+1} \mid E_1 \cap \ldots \cap E_j] \):

- All \( k C_S \) edges still remain (since we assume \( E_1 \cap \ldots \cap E_j \)).
- How many edges have been removed? At least \( j \)
  Not exactly \( j \), as we might have contracted a “parallel edge” earlier on, which has the effect of removing more than one edge from the graph.
- How many vertices have been removed? Exactly \( j \)
  The graph now has \((n - j)\) “vertices”, and each must have degree \( \geq k \) (why?); hence the graph now has at least \( k \cdot (n - j)/2 \) edges overall.

Therefore

\[
\Pr[E_{j+1} \mid E_1 \cap \ldots \cap E_j] \geq 1 - \frac{2k}{(n - j)k} = 1 - \frac{2}{n - j}.
\]

Expanding \( \prod_{j=3}^n \left(1 - \frac{2}{j}\right) \), we have

\[
\prod_{j=3}^n \frac{j - 2}{j} = \frac{1}{3} \cdot \frac{2}{4} \cdot \frac{3}{5} \cdot \frac{4}{6} \cdots \frac{n - 4}{n - 2} \cdot \frac{n - 3}{n - 1} \cdot \frac{n - 2}{n} = \frac{2}{n(n - 1)}
\]

So the probability that a single “run” of KargerOneTrial generates a cut which is minimal for the original graph is at least \( \frac{2}{n(n - 1)} \).

Could be more in practice. (WHY?)
Karger’s contraction Algorithm - Analysis

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\prod_{j=3}^{n} \frac{j-2}{j} = \left( \frac{1}{3} \right) \left( \frac{2}{4} \right) \left( \frac{3}{5} \right) \cdots \left( \frac{n-4}{n} \right) \left( \frac{n-2}{n-1} \right) \left( \frac{n-1}{n} \right) = \frac{2}{n(n-1)}
\]

So the probability that a single “run” of KARGERONE TRIAL generates a cut which is minimal for the original graph is at least \( \frac{2}{n(n-1)} \).

Could be more in practice. (WHY?)

Karger’s contraction Algorithm - Repeated iterations

We can improve our result by running KARGERONE TRIAL many times, and returning the minimum of all the different cuts.

If we do \( k \) trials, the probability that none is a min cut is at most \( \left(1 - \frac{2}{n(n-1)}\right)^k \).

Karger’s contraction Algorithm - Repeated iterations

Wrapping up

- Probability tools used in our analysis were simple: we have used conditional probability iteratively:
  \[
  \Pr[\cap_{j=3}^{n-2} E_j] = \Pr[\cap_{j=3}^{n-2} E_j \mid E_1] \cdot \Pr[E_1] = \Pr[\cap_{j=3}^{n-2} E_j \mid E_1 \cap E_2] \cdot \Pr[E_1 \cap E_2] \cdot \Pr[E_1] \cdot \Pr[E_1] = \ldots
  \]
  (also used simple inequalities relating \( (1 + \frac{1}{n})^n \) and \( e \))

- No approximation guarantee - analysis does not address the quality of \( C_S \) when it fails to be optimum.
We have shown that for a particular cut $C$, the probability of finding $C$ is at least $\frac{2}{n(n-1)}$. This implies that $|\mathcal{C}| \leq \frac{n(n-1)}{2}$, where $\mathcal{C}$ is the set of all Min-Cut.

Let $F_C$ be the event of finding $C$. For $C \neq C'$, $\Pr[F_C \cap F_{C'}] = 0$.

Thus, $\Pr[\cup_{C \in \mathcal{C}} F_C] = \sum_{C \in \mathcal{C}} \Pr[F_C] \leq 1$.

On the other hand, $\sum_{C \in \mathcal{C}} \Pr[F_C] \geq \sum_{C \in \mathcal{C}} \frac{2|\mathcal{C}|}{n(n-1)} = \frac{2|\mathcal{C}|}{n(n-1)}$.

Thus, $|\mathcal{C}| \leq \frac{n(n-1)}{2}$.

A u.a.r. cut is minimum with probability $\frac{|\mathcal{C}|}{2^{n-1}} \leq \frac{n(n-1)}{2^{2n-1}} = O\left(\frac{n^2}{2^n}\right)$. Hence Karger’s algorithm succeeds with probability exponentially higher than a random cut.
# Min-Cut

We have shown that for a particular cut $C$, the probability of finding $C$ is at least $\frac{2}{n(n-1)}$. This implies that $|C| \leq \frac{n(n-1)}{2}$, where $C$ is the set of all Min-Cut.

Let $F_C$ be the event of finding $C$. For $C \neq C'$, $\Pr[F_C \cap F_{C'}] = 0$.

Thus, $\Pr[\bigcup_{C \in C} F_C] = \sum_{C \in C} \Pr[F_C] \leq 1$.

On the other hand, $\sum_{C \in C} \Pr[F_C] \geq \sum_{C \in C} \frac{2}{n(n-1)} = \frac{2|C|}{n(n-1)}$.

Thus, $|C| \leq \frac{n(n-1)}{2}$.

A u.a.r. cut is minimum with probability $\frac{|C|}{2^n} \leq \frac{n(n-1)}{2^{2n-1}} = O\left(\frac{1}{n^2}\right)$. Hence Karger’s algorithm succeeds with probability exponentially higher than a random cut.

This is tight! Consider a cycle of length $n$.

## Other use of random contraction

Each edge fails with prob. $p$ independently. Can we compute or approximate $\Pr[G_p \text{ is connected}]$?

Exact evaluation is $\#P$-complete (Valiant, 1979), so a polynomial-time algorithm is unlikely.

Based on random contractions, Karger (1999) gave the first polynomial-time randomised approximation algorithm for Unreliability, namely $1 - \Pr[G_p \text{ is connected}]$.

The first efficient algorithm for Reliability was found by G. and Jerrum (2018). However it is based a variant of the constructive version of the Lovász Local Lemma.

### Expectation vs. Whp

There are two typical kinds of guarantees we will work with.

- **Expectation.** This includes the expected running time, expected output, etc.

- **With high probability (whp or w.h.p.).** The meaning of this can vary. Sometimes it means probability $1-o(1)$, which for example would include $1-\frac{1}{\log n}$. Sometimes it is stronger, namely $1-n^{-c}$.  

### Reading

Our next topic will be the “Coupon Collector” problem.

- Some of you may have seen the “Coupon Collector” problem in lower level classes.

- We will re-visit it, but as well as deriving the expected value, we will also bound the variance (2nd moment), and look at the implications of that.

- You might want to read sections 2.3, 2.4 and 3.3 of [MU] in advance (if your probability is rusty, also read 2.1, 2.2, 3.1 and 3.2)