Randomness and Computation

or, “Randomized Algorithms”

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The Monte Carlo Method

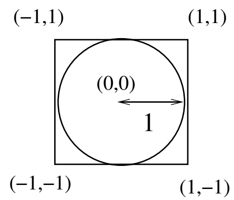
We’ve already met the concept of a Monte Carlo Algorithm*, which uses randomness during its computation to compute a value which is an* approximation *to the correct answer (satisfying some approximation guarantee, with high probability)*

The Monte Carlo Method *is a method for estimating values which exploits a relationship between (approximate) counting and (almost-uniform) sampling.*

The Monte Carlo Method

- Most common scenario for the Monte Carlo Method arises when the value we want to estimate is *the count of the number of combinatorial structures* satisfying a given criterion.
- We will usually rely on a close relationship between the problem of *counting the number of combinatorial structures* and *sampling one of the structures uniformly at random*.
  - Of course, in this setting, “the set of structures" means the set of structures according to some input. With *contingency tables* this input would be the number of rows m, and the number of columns n, and the specific lists of row sums $r = (r_1, \ldots, r_m)$ and column sums $c = (c_1, \ldots, c_n)$.
- A Markov chain can sometimes be employed to do the sampling.
- Other example “count the different combinatorial structures" include the set of proper k-colourings (of a given input graph $G = (V, E)$), the number of satisfying assignments (of a given DNF formula ϕ), etc.

Monte Carlo Method - cute example



Suppose we live in a world where π’s value is unknown. We estimate:

Algorithm ESTIMATEPI(m)

1. *count* $\leftarrow 0$
2. **for** $i \leftarrow 1$ **to** m
3. draw (X, Y) uniformly at random from the square
   ie draw each of X, Y *uniformly at random from the continuous distribution on* $[-1, 1]$
4. **if** $X^2 + Y^2 \leq 1$ **then**
5. *count* $\leftarrow$ *count* $+ 1$
6. **return** $\frac{4 \cdot count}{m}$

Monte Carlo Method - cute example

Can let Z_i be the indicator variable for the "i-th" (X, Y) lying inside the circle. Then for $Z = \sum_{i=1}^{m} Z_i$,

$$\mathrm{E}[Z] \;=\; \sum_{i=1}^{m} \mathrm{E}[Z_i] \;=\; m\frac{\pi \cdot 1^2}{2^2} \;=\; \frac{\pi m}{4}.$$

Hence $Z' = \frac{4Z}{m}$ is an estimate for the unknown value π.

Better estimate the higher m is. By Chernoff (4.6) if we have m samples, then for arbitrary $\epsilon \in (0, 1)$,

$$\begin{aligned} \Pr[|Z' - \mathrm{E}[Z']| \geq \epsilon\pi] &= \Pr\left[\left|Z - \frac{\pi m}{4}\right| \geq \frac{\epsilon\pi m}{4}\right] \\ &= \Pr[|Z - \mathrm{E}[Z]| \geq \epsilon \mathrm{E}[Z]] \\ &\leq 2e^{-\epsilon^2 \pi m/12}. \end{aligned}$$

Monte Carlo Method - cute example

Definition (Definition 11.1)

A randomized algorithm for estimating a (positive) quantity V (usually depending on certain input parameters) is said to give an (ϵ, δ) approximation if its output X satisfies

$$\Pr[|X - V| \leq \epsilon V] \geq 1 - \delta.$$

We know that for given $\epsilon \in (0, 1)$, that if we take m samples, then Algorithm EstimatePi gives an

$$(\epsilon, 2e^{-\epsilon^2 \pi m/12})$$

approximation.

We need $2e^{-\epsilon^2\pi m/12} \leq \delta$, equivalent to having $e^{-\epsilon^2\pi m/12} \leq \frac{\delta}{2}$, equivalent to having $\frac{\epsilon^2 \pi m}{12} \geq \ln(\frac{2}{\delta})$, equivalent to $m \geq \frac{12\ln(\frac{2}{\delta})}{\pi\epsilon^2}$.

Monte Carlo Method

Theorem (Theorem 11.1)

Let $X_1, \ldots, X_m$ be independent and identically distributed indicator random variables (ie Bernoulli with a fixed parameter), and $\mu = \sum_{i=1}^{m} \mathrm{E}[X_i]$. Then if $m \geq \frac{3\ln(\frac{2}{\delta})}{\epsilon^2\mu}$, we have

$$\Pr\left(\left|\frac{1}{m}\sum_{i=1}^{m} X_i - \mu\right| \geq \epsilon\mu\right) \leq \delta.$$

So for this m, sampling gives a (ϵ, δ)-approximation of μ.

Definition (Definition 11.2)

A *fully polynomial randomized approximation scheme (FPRAS)* for a problem is a randomized algorithm for which, given an input x and any parameters ϵ, δ with $0 < \epsilon, \delta < 1$ the algorithm outputs an (ϵ, δ)-approximation to the true value $V(x)$ in time polynomial in $\frac{1}{\epsilon}$, in $\ln(\frac{1}{\delta})$ and in the size of x.

Monte Carlo Method

> *The Monte Carlo Method involves taking a sequence of independent and identical samples $X_1, \ldots, X_m$ such that $\mathrm{E}[X_i] = V$, with m set large enough (see Theorem 10.1) to guarantee we have an (ϵ, δ)-approximation.*

The book discusses the reasons for using the Monte Carlo method. They discuss the situation of wanting to find "approximate" solutions for computational problems which are NP-hard to solve exactly (don't believe that NP-hard problems have polynomial-time algorithms).

More common in fact is the use of Monte Carlo in approximating the "count" of $\sharp P$-complete ("hard to count exactly in polynomial time") problems like proper k-colourings, contingency tables, etc. These will be from situations where the *decision problem* ("finding one") is polynomial-time.

The DNF counting problem

An alternative *normal form* for propositional logical formulae is *Disjunctive Normal Form (DNF)*, where each *clause* is now a *conjunction* ($\wedge$) or literals, and we have disjunctions ($\vee$) at the top-level. For example:

$$(x_1 \wedge \bar{x}_2 \wedge x_3) \vee (x_2 \wedge x_4) \vee (\bar{x}_1 \wedge x_3 \wedge x_4).$$

We are interested in *counting the number of satisfying assignments* to a given DNF formula.

- It is NP-hard to compute the *exact* number of satisfying assignments for a DNF, as this would solve the (NP-hard) problem of SAT (we can easily construct a DNF for the negation of the SAT formula ϕ, which has 2^n satisfying assignments $\Leftrightarrow \phi$ was unsatisfiable).
- Hence counting DNF assignments is $\sharp P$-complete.
- However, a DNF usually has *some/many* satisfying assignments, and we aim to approximately count.

The DNF counting problem - Naïve Approach

For a given DNF formula F over n variables, let $c(F)$ denote the *number of satisfying assignments to* $x_1, \ldots, x_n$.

$c(F)$ will be 0 *only if* it is the case that *every clause* contains x_i and $\bar{x}_i$ for some i. Easy to notice this before we start. We eliminate any of these *definitely unsatisfiable* clauses before we start.

Naïve approach to counting DNF assignments is to sample m *uniform random assignments* to $x_1, \ldots, x_n$ (from the set $\{0,1\}^n$) and check whether F is satisfied for each sample. The random variable X_i will be 1 if the i-th trial satisfies F, 0 otherwise). Then we estimate the fraction of these to satisfy F as $\frac{\sum_{i=1}^m X_i}{m}$, then we return

$$2^n \frac{\sum_{i=1}^m X_i}{m}$$

as the estimate of the total number of satisfying assignments.

The DNF counting problem - Naïve Approach

In order for

$$2^n \frac{\sum_{i=1}^m X_i}{m}$$

to be an $(\epsilon, \delta) -$ *approximation* for $c(F)$, we require that we have

$$\begin{aligned} \left|2^n \frac{\sum_{i=1}^m X_i}{m} - c(F)\right| &\leq \epsilon \cdot c(F) \quad \text{which happens} \Leftrightarrow \\ \left|\sum_{i=1}^m X_i - \frac{mc(F)}{2^n}\right| &\leq \epsilon \cdot \frac{mc(F)}{2^n} \end{aligned}$$

and by Chernoff this holds $\Leftrightarrow$ we have

$$m \geq \frac{3 \cdot 2^n \ln(\frac{2}{\delta})}{\epsilon^2 c(F)}.$$

But if it is the case that $c(F)$ is much much smaller than 2^n, then we need a huge number of samples (logical ... needle in haystack).

The DNF counting problem

Problem with using the Naïve Monte Carlo method is that it is infeasible (for any application) if the number of solutions is a small fraction of the sampled set.

For DNF this happens (say) when we have a small number of very large clauses. A random assignment is very unlikely to hit the good assignments.

On Friday, we will see a Monte Carlo algorithm which incorporates knowledge about “satisfying assignments per clause" to give an FPRAS for DNF.

Reading and Doing

Reading

- Sections 11.1, 11.2 from the book (11.2 is prep for Tuesday 19th).
- We will continue with the DNF counting problem on Tuesday.

Doing

- Exercises 11.3, 11.4 from the book.