Randomness and Computation

or, “Randomized Algorithms”

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Logical Formulae and the “satisfiability” question

Definition

Suppose we have a collection of (propositional) logical variables $x_1, \ldots, x_n$ for varying n.

A *literal* is any expression which is either x_i or $\bar{x}_i$, for some $i \in [n]$.

A *clause* is any *disjunction* of a number of literals.

We say a propositional formula $\phi : \{0,1\}^n \to \{0,1\}$ is in *Clausal Normal Form (CNF)* if it is of the form

$$C_1 \wedge C_2 \ldots \wedge C_h,$$

where every C_j is a *clause*.

The formula $\phi : \{0,1\}^n \to \{0,1\}$ is in *k-CNF* if it is in CNF and every clause contains *exactly* k literals.

The *SAT problem*, *k-SAT problem* is the problem of examining a given CNF (or k-CNF) expression and deciding whether or not it has a *satisfying assignment*.

Examples of SAT, k-SAT

Example of a SAT question:

$$(x_1 \vee x_8 \vee \bar{x}_6) \wedge (\bar{x}_4 \vee \bar{x}_7) \wedge (x_5 \vee x_7 \vee x_4 \vee x_2).$$

- For the formula above, easy to see there is a (many) satisfying assignment(s) to the x_i variables (any with $x_1 = 1, x_4 = 0, x_2 = 1$ would do, for example).
- In general, the *SAT* problem is NP-complete (we believe there is no polynomial-time algorithm).

Example of a 2-SAT question:

$$(x_1 \vee \bar{x}_2) \wedge (\bar{x}_1 \vee \bar{x}_3) \wedge (x_1 \vee x_2) \wedge (x_4 \vee \bar{x}_3) \wedge (x_4 \vee \bar{x}_1).$$

- There is a *polynomial-time* algorithm (either *randomized*, as we see today, or *deterministic*) to solve 2-SAT.
- The 3-SAT problem, and k-SAT for all $k > 3$, are all NP-complete.

2-SAT Randomized Algorithm

We will design a simple *randomized algorithm* for 2-SAT, and analyse its performance by analogy to a *Markov chain*.

Algorithm 2SATRANDOM(n; $C_1 \wedge C_2 \wedge \ldots \wedge C_\ell$)

1. Assign *arbitrary* values to each of the x_i variables.
2. $t \leftarrow 0$
3. **while** ($t < 2mn^2$ **and** some clause is unsatisfied) **do**
4. Choose an *arbitrary* C_h from all unsatisfied clauses;
5. Choose one of the 2 literals in C_h *uniformly at random* and flip the value of its variable;
6. **if** (we end with a satisfying assignment) **then**
7. **return** this assignment to the $x_1, \ldots x_n$ **else**
8. **return** FAILED.

Note that *arbitrary* is very different from *random*.

2-SAT Randomized Algorithm

Imagine Algorithm 2SATRANDOM running on our 2SAT example, with the initial assignment being $x_i = 0$ for all $i \in [n]$.

$$(x_1 \vee \bar{x}_2) \wedge (\bar{x}_1 \vee \bar{x}_3) \wedge (x_1 \vee x_2) \wedge (x_4 \vee \bar{x}_3) \wedge (x_4 \vee \bar{x}_1).$$

- Then $(x_1 \vee x_2)$ is the sole unsatisfied clause.
- Flipping the value of x_2 (say) from 0 to 1, will ensure that $(x_1 \vee x_2)$ now becomes satisfied.
- However, making this flip would *also* change the assignment for $(x_1 \vee \bar{x}_2)$, making this clause now *unsatisfied*.
- This is a balanced consequence overall (number of satisfied clauses stays the same). Note that a similar scenario would arise had we instead flipped x_1 to satisfy $(x_1 \vee x_2)$ (we would have violated $(x_4 \vee \bar{x}_1)$ in that case).

  However, there are examples where a flip might end up violating **many** *clauses.* So it's not so helpful for us to use "number of clauses satisfied" as our measure of progress.

2-SAT Randomized Algorithm - Analysis

Consider an (unknown so far) satisfying assignment $S \in \{0,1\}^n$ that makes our 2SAT formula ϕ true (satisfies all the clauses).

Our "measure of progress" will be *the number of indices k such that $x_k = S_k$*, $(x_1, \ldots, x_n)$ being the current assignment.

We will analyse the *expected number of steps* before $(x_1, \ldots, x_n)$ becomes S.

- This of course assumes the formula ϕ has some satisfying assignment.
- Of course we really have $(x_1^t, \ldots, x_n^t)$ (for time step t), as the assignment changes as we proceed.
- Note that if ϕ does not have any satisfying assignment, Algorithm 2SATRANDOM always returns FAILED (as it should)

2-SAT Randomized Algorithm - Analysis

To analyse the behaviour of Algorithm 2SATRANDOM when given a 2CNF formula ϕ that *is* satisfiable, we need some definitions.

Definition

For our given satisfiable 2SAT formula ϕ, let S be some satisfying assignment for ϕ.

Let $(x_1^t, \ldots, x_n^t)$ denote the assignment to the logical variables after the t-th iteration of the loop at 3.

Let X_t denote the number of variables of the assignment $(x_1^t, \ldots, x_n^t)$ having the same value as in S.

We work with the X_t variable mainly, and bound the time before it reaches the value n.

2-SAT Randomized Algorithm - Analysis

Some observations:

- If X_t ever hits the value 0, and ϕ is not yet satisfied, we are guaranteed that at the next step, $X_{t+1} = 1$.

$$\Pr[X_{t+1} = 1 \mid ((X_t = 0)\ \&\ \phi \text{ not-sat})] \;=\; 1.$$

- Alternatively, suppose $X_t = j$ for some value $j \in \{1, \ldots, n-1\}$ and that ϕ is unsatisfied.

  Then on any of the currently unsatisfied clauses, we know the current assignment x^t must differ from S on *at least one* of the two variables. Hence *with probability at least 1/2*, we will increase the value of X_t by 1 (and with probability at most $1/2$ decrease the value of X_t by $1/2$).

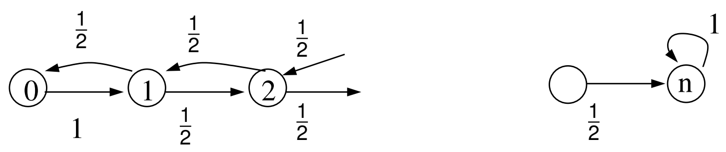
$$\Pr[X_{t+1} = j+1 \mid ((X_t = j)\ \&\ \phi \text{ not-sat})] \;\geq 1/2;$$
$$\Pr[X_{t+1} = j-1 \mid ((X_t = j)\ \&\ \phi \text{ not-sat})] \;\leq 1/2.$$

2-SAT Randomized Algorithm - Analysis

We want to imagine the *progress of* 2SATRANDOM as a Markov chain on the states $0, 1, \ldots, n$. Our concern is bounding the *expected number of steps for X_t to hit the state n* (from an *arbitrary* starting point).

- Markov chains should be *memoryless*, and this is problematic.
- The value for $\Pr[X_{t+1} = j + 1 \mid ((X_t = j)\ \&\ \phi \text{ not-sat})]$ can be 1/2 or 1 depending on *how many variables of the chosen clause currently disagree with S*. This may have been affected by *earlier* flips done by the algorithm.
- We choose to "tweak" the probabilities and study the process on $\{0, 1, \ldots, n\}$ where we have to make the process memoryless. We consider a slightly different process on $\{0, 1, 2, \ldots, n\}$ defined by the variable Y_t on the next slide.

2-SAT Randomized Algorithm - Analysis



The Markov chain Y_t

Consider the Markov chain $Y_0, Y_1, \ldots, Y_t, \ldots$ such that

$$\begin{aligned} Y_0 &= X_0; \\ \Pr[Y_{t+1} = 1 \mid ((Y_t = 0)\ \&\ \phi \text{ not-sat})] &= 1; \\ \Pr[Y_{t+1} = j + 1 \mid ((Y_t = j)\ \&\ \phi \text{ not-sat})] &= 1/2; \\ \Pr[Y_{t+1} = j - 1 \mid ((Y_t = j)\ \&\ \phi \text{ not-sat})] &= 1/2. \end{aligned}$$

Clearly the *expected number of steps for X_t to hit n* is $\leq$ that for Y_t.

2-SAT Randomized Algorithm - Analysis

For any $j = 0, \ldots, n - 1$, define h_j to be the *expected number of steps to hit n starting from j*.

- h_j is the $h_{j,n}$ measure from lecture 14 (we omit n because we have the same target for each j);
- Clearly, the expected number of steps for 2SATRANDOM to find a satisfying assignment is *at most* $\max_j h_j$ (may well be better).
- We will bound h_j for every $j = 0, 1, \ldots, n$.

2-SAT Randomized Algorithm - Analysis

We have $h_n = 0$ and $h_0 = h_1 + 1$ for the "end cases".

We will use Z_j, for $0, 1, \ldots, n - 1$, to be the random variable for the "number of steps" to reach n from j (h_j will be $\mathrm{E}[Z_j]$).
For $j = 1, \ldots, n - 1$, recalling the steps of the "random walk", and using linearity of expectation:

$$\begin{aligned} \mathrm{E}[Z_j] &= \frac{1}{2}(\mathrm{E}[Z_{j-1}] + 1) + \frac{1}{2}(\mathrm{E}[Z_{j+1}] + 1), \\ h_j &= \frac{1}{2}(h_{j+1} + 1 + h_{j-1} + 1) \end{aligned}$$

This gives us the following system of equations:

$$\begin{aligned} h_0 &= h_1 + 1 \\ h_j &= \frac{h_{j-1} + h_{j+1}}{2} + 1 \qquad \text{for } j = 1, \ldots, n - 1 \\ h_n &= 0 \end{aligned}$$

2-SAT Randomized Algorithm - Analysis

We show by induction that for $j = 0, \ldots, n-1$,

$$h_j = h_{j+1} + 2j + 1.$$

Proof.

Base case: If $j = 0$, $2j + 1 = 1$, and we were given $h_0 = h_1 + 1$.
Inductive step: Suppose this was true for $j = k - 1$ (we had $h_{k-1} = h_k + 2(k-1) + 1$, this is our (IH)). Now consider $j = k$.
By the "middle case" of our system of equations,

$$\begin{aligned} h_k &= \frac{h_{k-1} + h_{k+1}}{2} + 1 \\ &= \frac{h_k + 2(k-1) + 1}{2} + \frac{h_{k+1}}{2} + 1 \qquad \text{by our (IH)} \\ &= \frac{h_k}{2} + \frac{h_{k+1}}{2} + \frac{2k+1}{2} \end{aligned}$$

Subtracting $\frac{h_k}{2}$ from each side, this is equivalent to

$$h_k = h_{k+1} + 2k + 1,$$

as claimed.

2-SAT Randomized Algorithm - Analysis

Lemma (Lemma 7.1)

Assume that the given 2CNF formula has a satisfying assignment, and that 2SATRANDOM is allowed to carry out as many iterations as it wants to find a satisfying assignment. Then the expected number of iterations of 3. to find that assignment is at most n^2.

Proof.

We showed that the expected number of iterations is at most $\max_{j=0,\ldots,n-1}\{h_j\}$. We now know the max is h_0.
Applying $h_k = h_{k+1} + 2k + 1$ iteratively, we have

$$\begin{aligned} h_0 &= \sum_{k=0}^{n-1}(2k+1) + h_n \\ &= 2\sum_{k=0}^{n-1} k + n + 0 \\ &= 2\frac{(n-1)n}{2} + n = n^2. \end{aligned}$$

Probability of failure

Theorem

Algorithm 2SATRANDOM is parametrized by m, and the algorithm will perform up to $2mn^2$ iterations of the loop.
Then, when there is a satisfying assignment for ϕ, the probability that 2SATRANDOM does not discover one, is at most 2^{-m}.

Proof.

We use Markov's Inequality, but not "all-in-one" (which would only bound our failure below $2^{-1}m^{-1}$,
Instead we group the $2mn^2$ iterations into m "blocks" of $2n^2$ each, and Markov gives failure 2^{-1} for an individual block. Hence failure overall is at most $(2^{-1})^m = 2^{-m}$. □

Reading and Doing

Reading

- This material is from Section 7.1 of [MU].
- Section 7.4 from the book is interesting (we were looking at a random walk on the line today).

Doing

- week 11 tutorial sheet.
- Exercise 7.10 from [MU] requires similar ideas to those used to prove the result for 2-SAT ... but quite a challenge to get all details right.