Randomness and Computation
or, “Randomized Algorithms”

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The Probabilistic Method

The Probabilistic Method is a nonconstructive method of proof, primarily used in combinatorics and pioneered by Paul Erdős, for proving the existence of a desired kind of mathematical object. It works by showing that if we randomly choose objects from a specified class, the probability that the result has the desired property is greater than zero. This is enough to tell us that there must be at least one object with the desired property in the class.

Note that although this approach uses probability, the result (that some object with the property exists) will be definite, not “in probability”.

Slightly different theme to the rest of the results in this course, as we are concerned with showing existence (rather than constructing the object). However, sometimes we can derandomize/construct.
Graphs and Colourings

A common concept in graph theory is the concept of a colouring of a graph. If we have \( k \) different colours, we usually identify them with the set \( \{1, \ldots, k\} \).

- We can consider the different ways of colouring the vertices of a graph \( G = (V,E) \) with those \( k \) colours.
  - A \textit{k-colouring} is any assignment \( c : V \rightarrow \{1, \ldots, k\} \) of colours to vertices (every \( v \in V \) gets some colour \( c(v) \)).
  - A \textit{proper k-colouring} is any \( c : V \rightarrow \{1, \ldots, k\} \) such that for every \( e = (u,v), e \in E \), we have \( c(u) \neq c(v) \).
  - For a given graph \( G = (V,E) \), it is often of interest to ask \textit{what is the minimum } \( k \text{ needed to properly colour } G \). For sure, we require \( k \leq \text{“max degree of } G +1\text{“} \).
  - Lots of research effort have gone into polynomial-time algorithms to approximate (exact is NP-hard) the minimum \( k \) for a given \( G \). \textbf{Not our concern today}

- Alternatively we can consider the different ways of colouring the edges of a graph \( G = (V,E) \).
Our example - Ramsay numbers

Our focus today is 2-colouring the **edges** of the complete graph $K_n$.

- $K_n$ is the **complete graph** on $n$ vertices (for every $i, j \in [n], i \neq j$, we have the edge $(i,j)$).

- We are interested in 2-colourings of $K_n$, every edge “blue” or “red”. (Clearly this cannot be a proper (edge) colouring if $n \geq 3$).

- Our concern is whether we can colour $K_n$ with our two colours and make sure that we do not have any “all-blue” or “all-red” subgraph which is “too large”.

- The **“Ramsay number”** $R(k,k)$ is the smallest value for $n$ such that in any **two-colouring** of the edges of $K_n$, there must **either** be a red $K_k$ (“all-red” of size $k$) **or** a blue $K_k$ (“all-blue” of size $k$).

The value of $R(k,k)$ increases with $k$.

**class:** What is $R(2,2)$? And $R(3,3)$ (board)?
Lower Bound on $R(k, k)$

We prove an *lower bound* on $R(k, k)$ for general $k$. This was first shown by Erdős in 1947.

**Theorem (Theorem 6.1)**

Consider $R(k, k)$ for some $k \geq 2$. For any $n$ such that $\binom{n}{k}2^{1 - \binom{k}{2}} < 1$, we have $R(k, k) > n$.

**Proof.**

Write down the *expected* number of “all red”/“all blue” $K_k$ subgraphs, when the edges of $K_n$ are coloured uniformly at random by red/blue. For a particular $K_k$ subgraph, probability of being *monochromatic* is $2 \cdot 2^{-\binom{k}{2}} = 2^{1 - \binom{k}{2}}$.

There are $\binom{n}{k}$ different $K_k$ subgraphs to consider in $K_n$.

The *expected number of monochromatic subgraphs* of $K_n$ is therefore

$$\binom{n}{k} \frac{2}{2^{\binom{k}{2}}}.$$
Lower Bound on $R(k, k)$

**Theorem (Theorem 6.1)**

Consider $R(k, k)$ for some $k \geq 2$. For any $n$ such that $\binom{n}{k} 2^{1 - \binom{k}{2}} < 1$, we have $R(k, k) > n$.

**Proof cont’d.**

Now if $\binom{n}{k} \frac{2^{\binom{k}{2}}}{\binom{k}{2}} < 1$ (as per the conditions), this implies that the expected number of monochromatic $K_k$ subgraphs is less than 1 when $K_n$’s edges are randomly two-coloured.

Hence there must be at least one two-colouring of $K_n$’s edges without any monochromatic $K_k$ subgraph.

So the Ramsey number $R(k, k)$ is larger than any such $n$.

To be guaranteed a monochromatic $K_k$ we need $\binom{n}{k} \geq 2^{\binom{k}{2} - 1}$
Lower Bound on $R(k, k)$

**Corollary**

If $k \geq 3$, then for $R(k, k) > \lfloor 2^{k/2} \rfloor$.

**Proof.**

Just algebraic manipulation.

Consider $\binom{n}{k} \cdot 2^{1-(\frac{k}{2})}$ for the given value of $n = \lfloor 2^{k/2} \rfloor$. This is

$$\frac{n \cdots (n-k+1)}{k!} \cdot 2^{1-(\frac{k}{2})}$$

$$< \frac{n^k}{k!} \cdot 2^{1-\frac{k(k-1)}{2}}$$

$$= \frac{n^k}{k!} \cdot 2^{1+\frac{k}{2}} 2^{-\frac{k \cdot k}{2}}$$

$$= \frac{n^k}{2^{\frac{k^2}{2}}} \cdot \frac{2^{1+\frac{k}{2}}}{k!}$$

$$= \left(\frac{n^k}{2^{\frac{k^2}{2}}}\right)^k \cdot \frac{2^{1+\frac{k}{2}}}{k!}$$

$$< 1 \cdot 1,$$

as required.
Making this method constructive ("derandomization")

In the proof of Theorem 6.1 about random edge-colourings of $K_n$ and the presence of any monochromatic $K_k$s, we focused on the situation when we have $\binom{n}{k} 2^{1-\binom{k}{2}} < 1$. However, we showed . . .

**Corollary**

*Let $k \geq 2$. Then for any complete graph $K_n$, the expected number of monochromatic $K_k$ subgraphs in a uniform random 2-colouring of the edges of $K_n$ is at most $\binom{n}{k} 2^{1-\binom{k}{2}}$.***

**Corollary**

*Let $k = 4$. Then for any complete graph $K_n$, the expected number of monochromatic $K_4$ subgraphs in a uniform random 2-colouring of the edges of $K_n$ is at most $\binom{n}{4} 2^{-5}$.***

Remember the various $K_k$ copies we consider are not necessarily disjoint, expectation is taken over all of them.
Making this method constructive ("derandomization")

Using the second Corollary on slide 8, if the expectation is at most \( \binom{n}{4} 2^{-5} \) (over all random 2-colourings), then there is some specific 2-colouring of \( K_n \) that has \( \leq \binom{n}{4} 2^{-5} \) monochromatic \( K_k \) copies.

We can construct a specific 2-colouring to satisfy this using the method of conditional expectation (and "deferred decisions").

The idea:

- Let \( f \) be a specific edge of \( K_n \).
- A random 2-colouring has probability \( \frac{1}{2} \) of setting \( d \) blue, and probability \( \frac{1}{2} \) of setting \( e \) red.
- The colours of all the other edges are set uniformly and independently with probability \( \frac{1}{2} \).
- Hence, for at least one of the events \( c(f) = \text{red} \), \( c(f) = \text{blue} \), the (conditional) number of expected monochromatic \( K_k \) is \( \leq \binom{n}{4} 2^{-5} \).
- Find a way of determining this colour for \( f \), and iterate.
Making this method constructive ("derandomization")

**Theorem**

*For every integer $n$, we can construct a specific 2-colouring of $K_n$ such that the expected number of monochromatic $K_4$ subgraphs is at most $\left(\binom{n}{4}\right)2^{-5}$.*

**Proof.**

To help with the construction, we define a *weight function* $w$ on copies of $K_4$ which will allow us to measure the expected "value" of colouring particular edges blue or red.

Suppose we are part-way through the construction, and some (but not all) edges have their colour fixed.

- We have some partial colouring $c : F \rightarrow \{\text{blue, red}\}$, where $F \subseteq E(K_n)$.

- We maintain the invariant that the expected number of monochromatic $K_k$ copies, taken over the remaining random 2-colourings for the edges in $E(K_n) \setminus F$, is $\leq \left(\binom{n}{4}\right)2^{-5}$. 
Making this method *constructive* ("derandomization")

**Theorem**

*For every integer $n$, we can construct a specific 2-colouring of $K_n$ such that the expected number of monochromatic $K_4$ subgraphs is at most $\binom{n}{4}2^{-5}$.***

**Proof cont’d.**

The weight function $w$ assigns a non-negative value to every subgraph $K$ which is a copy of $K_4$ in $K_n$. Let $c(K)$ be the set of colours already seen on edges of $K$, at this stage of the partial colouring. Define

$$w(K) = \begin{cases} 
0 & \text{if } c(K) = \{\text{blue, red}\}. \\
2^{-5} & \text{if } c(K) = \emptyset \text{ (all edges uncoloured).} \\
2r^{-6} & \text{if } |c(K)| = 1, \text{ and } r \text{ of } K'\text{’s edges have this colour}
\end{cases}$$

The *total weight* of the partially coloured $K_n$ is

$$W_F = \sum_{\text{K a } K_4 \text{ copy in } K_n} w(K).$$
Making this method constructive ("derandomization")

Theorem
For every integer $n$, we can construct a specific 2-colouring of $K_n$ such that the expected number of monochromatic $K_4$ subgraphs is at most $\binom{n}{4}2^{-5}$.

Proof cont’d.
Note $w(K)$ is the probability of that particular $K$ becoming a monochromatic copy of $K_4$ in a uniform random 2-colouring of the edges $E(K_n) \setminus F$.

The expected number of monochromatic $K_4$ copies in a uniform random 2-colouring of the so-far uncoloured edges, is therefore equal to $W_F$.

To build our “good" 2-colouring, we start with a fixed ordering $e_1, \ldots, e_{n(n-1)/2}$ of the edges of $K_n$. $W_\emptyset$ is $\binom{n}{4}2^{-5}$. 
Making this method constructive ("derandomization")

Proof.

The Algorithm:

1. for $i \leftarrow 1$ to $n(n - 1)/2$ do
   
   ($F$ is $e_1, \ldots, e_{i-1}$, and these edges are coloured)

2. Calculate $W_{\text{red}}$, the effect on $W_F$ of colouring $e_i$ red.

3. Calculate $W_{\text{blue}}$, the effect on $W_F$ of colouring $e_i$ blue.

4. if $W_{\text{red}} < W_{\text{blue}}$ then Set $c(e_i) = \text{red}$, $W \leftarrow W_{\text{red}}$.

5. else Set $c(e_i) = \text{blue}$, $W \leftarrow W_{\text{blue}}$.

6. else Set $F \leftarrow F \cup \{e_i\}$.

- Note that the value of $W_F$ never increases through the iteration of this process. Hence we end up with a colouring $c$ which has fewer than $W_0 = \binom{n}{4}2^{-5}$ monochromatic $K_4$s.

- $e_i$ can belong to at most $n^2$ $K_4$s in $K$, so the $W_{\text{red}}$, $W_{\text{blue}}$ values can be calculated in $\Theta(n^2)$ time.
The theorem on slides 10-13 can be considered to be a “derandomization” of the result on expected number of monochromatic $K_4$s in $K_n$.

- We were able to use conditional expectation to construct a specific colouring with less than or equal to the expected number of monochromatics.
- Our algorithm was in fact polynomial-time (about $n^2$ iterations, each doing $\Theta(n^2)$ work, so roughly $\Theta(n^4)$).

- We can use the method of conditional probabilities to derandomize algorithms like Karger’s MIN-CUT algorithm from lectures 3-4.
Reading and Doing

Reading

▶ You will want to read Sections 6.1, 6.2, 6.3 from the book.
▶ Sections 6.4 and 6.5 also worth reading, though not required.
▶ The Theorem on derandomizing monochromatic $K_4$ is not in the book.

Doing

▶ Write down the details of how to derandomize Karger’s MIN-CUT algorithm so it always (deterministically) returns a cut of value $\leq \frac{|E|}{2}$. Some similarity to what’s in Section 6.2 (but “min" for us).
▶ Exercise 6.8 from the book (you’ll need Section 6.4).