Randomness and Computation
or, “Randomized Algorithms”

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new topic: The Probabilistic Method cont’d.

In Lecture 11 (continuing today) we saw the application of the
Probabilistic Method:

▶ To allow us to set bounds on certain parameters that will ensure
a randomly-drawn combinatorial object (from whatever pool of
possible objects we are focusing on) has some desirable
property with probability > 0.
(our example property was that the edge 2-colouring of \(K_n\) graph
would be without any monochromatic \(K_k\) subgraphs, assuming \(n\)
is large enough wrt a lower bound)
▶ The “probabilistic method” then allows us to infer that at
least one of the combinatorial objects (from our pool) must
have the desired property.
▶ Sometimes we can also derandomize this existence proof and
actually construct an object satisfying the desired property.
(need to have a de-composable (wrt deferred decisions) for
drawing the random object, then apply conditional probabilities)

The Probabilistic Method cont’d.

Previous slide refers to “having some desirable property with
probability > 0”.
In practice, often the approach will be to evaluate with an expectation
rather than a probability.
Many of the examples of the probabilistic method we meet in RC
involve showing that we can construct a combinatorial object (from
some pool) that avoids having some banned sub-structure.
▶ We can consider the expected number of the banned
substructures, when we draw an object from the sample pool;
▶ Sometimes it will be possible to evaluate the expected number
of banned-substructures. If this is < 1, then the probability that
there are some combinatorial objects that avoid all
banned-substructures is > 0.

Second Moment Method

Examples of the probabilistic method so far only worked with
expectation. If we had \(E[\cdot] < 1\) for the number of banned
sub-structures, we knew there must be some object that has none of
the banned sub-structures at all.
However, if we have \(E[\cdot] > 1\), things are less clear. There are
definitely objects containing the banned sub-structures, but how likely
they are is not clear.
Early on in the course we gave the definition of the second moment of
a discrete random variable \(X\), this being \(E[X^2]\). Then variance is
\(E[X^2] - E[X]^2\).
The second moment (with Chebyshev) can help us show that a
typical sample is likely to have \(X\) close to \(E[X]\).
Second Moment Method

Theorem (Theorem 6.7)
For any integer-valued random variable $X$ with positive expectation, we have

$$\Pr[X = 0] \leq \frac{\text{Var}[X]}{[E[X]]^2}.$$ 

Proof.
Really just a special case of Chebyshev’s Inequality. We are interested in $\Pr[X = 0]$, which is equal to $\Pr[E[X] - X = E[X]]$. Also,

$$\Pr[E[X] - X \geq E[X]] \leq \Pr[E[X] - X \geq E[X]] \leq \Pr[E[X] - X \geq E[X]].$$

Then this final $\Pr[\cdot]$ fits the form for Chebyshev’s Inequality with $a = E[X]$, so applying Chebyshev gives us

$$\Pr[|E[X] - X| \geq E[X]] \leq \frac{\text{Var}[X]}{[E[X]]^2},$$

and this right-hand side also bounds $\Pr[X = 0]$. 

RC (2018/19) – Lecture 12 – slide 5

Threshold for 4-cliques in $G_{n,p}$

Random model $G_{n,p}$, interested in whether $G \leftarrow G_{n,p}$ contains a 4-clique.

Theorem (6.8(a))
Suppose we have some probability sequence $p = p(n)$ such that $p(n) = o(n^{-2/3})$.
Then for any $\epsilon > 0$, for sufficiently large $n$, the graph $G \leftarrow G_{n,p}$ will contain a 4-clique with probability less than $\epsilon$.

Proof. Recall that $p = p(n), p(n) = o(n^{-2/3})$ means that for every $\delta > 0$, there is some $n_\delta \in \mathbb{N}$ such that $p(n) < \delta \cdot n^{-2/3}$ for all $n \geq n_\delta$.

Let $X$ denote the number of 4-cliques in $G \leftarrow G_{n,p}$.

Then $E[X] = E[\sum_{f \subseteq [n], |f| = 4} X_f]$, where $X_f = 1$ if those 4 vertices form a clique, 0 otherwise.

Linearity of exp. gives $E[X] = \sum_{f \subseteq [n], |f| = 4} E[X_f].$

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Threshold for 4-cliques in $G_{n,p}$

Proof of 6.8 (a) cont’d.
Now we compute $E[X_f]$ for a specific subset $f = \{u, v, w, x\}$.
These 4 vertices form a clique $\iff$ all 6 edges are in $G \leftarrow G_{n,p}$. This happens with probability $p^6$.

This value of $E[X_f]$ is independent of the particular $f$, and there are exactly $\binom{n}{4}$ subsets satisfying $f \subseteq [n], |f| = 4$. Hence

$$E[X] = \binom{n}{4} p^6 = \frac{n(n-1)(n-2)(n-3)}{24} \cdot p^6.$$ 

Now consider $\delta = \sqrt{24\epsilon}1/6$ in the definition of $o(n^{-2/3})$; then for $n \geq n_\delta$ we have $p \leq \delta \cdot n^{-2/3}$, then $p^6 \leq 24 \epsilon (n^{-2/3})^6 = 24 \epsilon n^{-4}$. Then

$$E[X] \leq \frac{n(n-1)(n-2)(n-3)}{24} \cdot p^6 \leq \epsilon.$$ 

Clearly the graph is more likely to have a 4-clique if $p$ has a higher value (and $G \leftarrow G_{n,p}$ is likely to have more edges).

We are interested in whether the drawn graph $G \leftarrow G_{n,p}$ contains a 4-clique or not.

We will show that there is a threshold for “$G \leftarrow G_{n,p}$ having a 4-clique” when $p$ is either side of $\Theta(n^{-2/3})$.

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Threshold for 4-cliques in $G_{n,p}$

We are interested in the random model $G_{n,p}$, where we draw a random graph on $n$ vertices by independently doing a Bernoulli trial for each potential edge $\{u, v\}, u \in V, v \in V \setminus \{u\}$, adding the edge with probability $p$, omitting that edge if the trial returns 0.

We are interested in whether the drawn graph $G \leftarrow G_{n,p}$ contains a 4-clique or not.

Clearly the graph is more likely to have a 4-clique if $p$ has a higher value (and $G \leftarrow G_{n,p}$ is likely to have more edges).

We will show that there is a threshold for “$G \leftarrow G_{n,p}$ having a 4-clique” when $p$ is either side of $\Theta(n^{-2/3})$.

RC (2018/19) – Lecture 12 – slide 6
Threshold for 4-cliques in $G_{n,p}$

Proof of 6.8(b) cont’d.
We want to calculate $\text{Var}[X]$, however, not all the $X_f$ variables are independent, so we can’t apply $\text{Var}[X] = \sum_{f \subseteq [n], |f|=4} \text{Var}[X_f]$.

We can apply the result from our early lectures to rewrite:

$$\text{Var}[X] \leq \mathbb{E}[X] + \sum_{f \subseteq [n], |f|=4} \left[ \sum_{g \subseteq [n], g \neq f, |g|=4} \text{Cov}[X_f, X_g] \right].$$

case (a): $|f \cap g| \leq 1$:
In this case $f$ and $g$ share no edges at all; and $\mathbb{E}[X_f] \mathbb{E}[X_g]$ is $\mathbb{E}[X_f] \mathbb{E}[X_g]$, hence $\text{Cov}[X_f, X_g] = 0$. Most likely case, there are $\binom{n}{4} + n \cdot \binom{n-1}{3} \binom{n-4}{3}$ pairs like this. But their contribution to the “double sum” is 0, we can ignore.

case (b): $|f \cap g| = 2$:
Then $f$ and $g$ share one edge, and $\mathbb{E}[X_f] \mathbb{E}[X_g] = \mathbb{E}[X_f] \mathbb{E}[X_g]$, and hence
$\text{Cov}[X_f, X_g] = \mathbb{E}[X_f] \mathbb{E}[X_g] - (\mathbb{E}[X_f])^2 \leq \mathbb{E}[X_f] \mathbb{E}[X_g]$. There are $\binom{n}{2} \binom{n-2}{2} \binom{n-4}{2}$ pairs like this.

Note that given $\epsilon > 0$,

Proof. Recall that $p = p(n)$, $p(n) = \omega(n^{-2/3})$ means that for every $\delta > 0$, there is some $n_0 \in \mathbb{N}$ such that $p(n) > \delta \cdot n^{-2/3}$ for all $n \geq n_0$.

Let $X$ denote the number of 4-cliques in $G \leftarrow G_{n,p}$.

We know that $\mathbb{E}[X] = \binom{n}{4} p^6$ and by a similar argument to before, if $n > n_0$ of $\omega(n^{-2/3})$ then $\mathbb{E}[X] = \binom{n}{4} \cdot p^6 = \omega(1) \to \infty$ as $n \to \infty$.

This means $\mathbb{E}[X]$; however, it doesn’t imply a lower bound for $\text{Pr}[X > 0]$; need to examine the second moment.

Threshold for 4-cliques in $G_{n,p}$

Proof of 6.8(b) cont’d.

Putting it all together . . .

$$\sum_{f \subseteq [n], |f|=4} \sum_{g \subseteq [n], g \neq f, |g|=4} \text{E}[X_f X_g] \leq (\binom{n}{3} \binom{n-3}{1}) p^8 + (\binom{n}{2} \binom{n-2}{2}) \binom{n-4}{2} p^{11} = O(n^4 \cdot p^8 + O(n^5 \cdot p^9 + O(n^6 \cdot p^{11})$$

Note that given $p = \omega(n^{-2/3})$, both of these terms is “little-o” of $\mathbb{E}[X]^2 = \Theta(p^{12} \cdot n^8)$.

Also adding $\mathbb{E}[X] = p^6 \binom{n}{4}$ to the “double sum”, this is also “little-o” of $\Theta(p^{12} \cdot n^8)$; hence $\text{Var}[X]$ is $o(p^{12} \cdot n^8)$, and applying Chebyshev we find that

$$\text{Pr}[X \geq 0] \leq \frac{o(p^{12} \cdot n^8)}{\Theta(p^{12} \cdot n^8)},$$

which tends to 0 as $n \to \infty$. 

Notes

Reading
- You will want to read Sections 6.4, 6.5, 6.6, 6.7 from the book.

Doing
- Tutorial sheet for 5th, 6th March (week 7).
- You will be getting the coursework 2 specification next week.