SAT-Solving: From Davis-Putnam to Zchaff and Beyond

Day 1: SAT Basics

Lintao Zhang

Microsoft Research

Automated Reasoning: Motivations

- As a curiosity of mathematicians and inventers
  - Demonstrator, Charles Stanhope, 1777
  - Logic Machine, William Stanley Jevons, 1869
- Artificial Intelligence and foundation of mathematics
  - Mechanical theorem proving
  - Reasoning on knowledge base
- Electronic Design Automation
  - ATPG
  - Logic synthesis
- Verification of digital systems
  - Equivalence checking
  - Model checking
  - Safety of programs, concurrent processes

How to Perform Automatic Reasoning?

- Modeling: Abstract the problem into logic
  - Boolean propositional logic
  - Temporal logic
  - Set theory
  - First order logic

- Proof: Use automatic decision procedures to determine the correctness (*validity*) of the resulting logic
  - SAT Solvers and BDDs
  - Model Checker
  - Theorem Provers

Propositional Logic

- Variable Domain: **True/False** or **1/0**
- Logic operations: **and** $\wedge$ $\cdot$**, or** $\vee$ **+, not** $\neg$ **’**
  - It’s also easy to express **Imply** $\rightarrow$**, equivalence** $\leftrightarrow$
- If a and b are Boolean, then these are propositional formulas:
  - $a \cdot b + a' \cdot c$
  - $1 \cdot a = 0$
  - $1 + a = 1$
- These are not propositional logic:
  - $3 + x = x + 3$; -- Integer domain
  - $\forall a \exists b\ (a+b)(a'+b')$ -- Quantifiers
  - If $a = b$ then $f(a)=f(b)$ -- Uninterpreted function
- It is the basis of all other logics.

What is SAT?

- Boolean Satisfiability (SAT).
- Operates on Boolean Propositional Logic
- Check if a complex logical relationship can ever be true (or satisfiable)
  - x OR y is true when x is true or y is true (satisfiable)
  - x AND (NOT x) can never be true (unsatisfiable)
- Tautology Checking
- Looks easy, but gets hard very quickly as the size of the problem increases
  - Size measured in terms of:
    - Number of variables
    - Number of operations

Why is SAT Important?

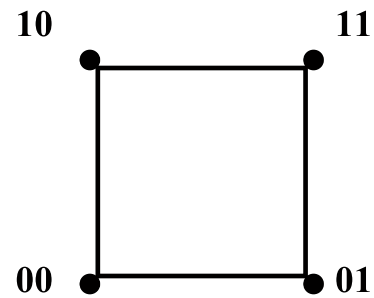
- Theoretical importance
  - It's the first NP-Complete problem discovered by Cook in 1971
- It's everywhere
  - Automatic Test Pattern Generation
  - Combinational Equivalence Checking
  - Bounded Model Checking
  - AI Planning
  - Theorem Proving
  - Software modeling and verification
  - ... ...
- We have powerful SAT solvers that can solve practical problems
  - SAT solving has been well studied for at least 40 years.
  - Recent breakthroughs make SAT solver highly efficient
    - Can handle over a million variables and operations
    - Seen wide use in the industry
  - Can we do better?

Course Schedule

- 3-day mini-course
  - Today: Basics of SAT solving
  - Tomorrow: Efficient Implementation of SAT solvers
  - Wednesday: Recent Developments in SAT research
- Emphasis on Engineering, not math or just algorithms
- Lectures in the morning, projects and discussion in the afternoon
- Main course project: Implementing an SAT solver
  - Require some knowledge of C/C++ and STL

Boolean n-Space

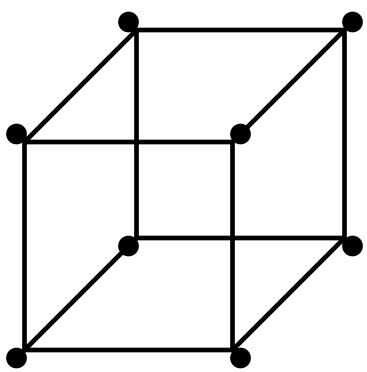
$B = \{0,1\}$



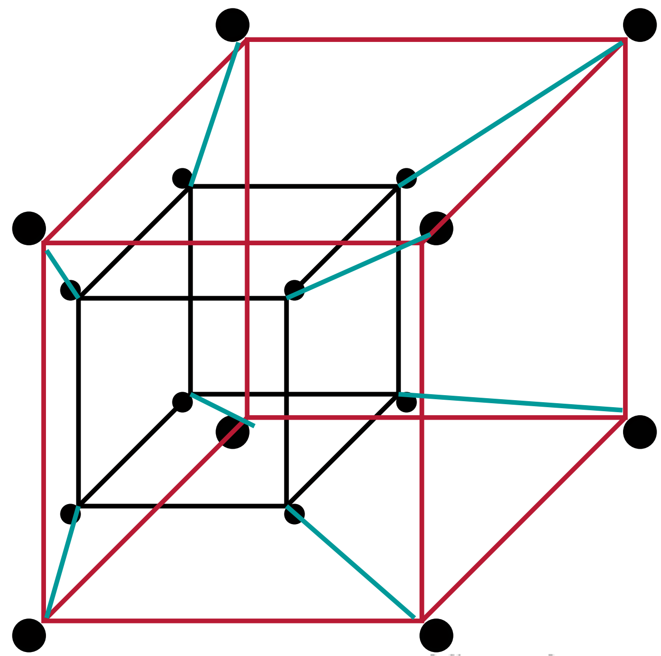
B^0

B^1

$B^2 = BxB$



B^3

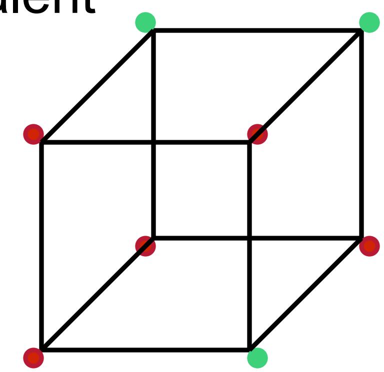


B^4

Boolean Functions

$f(x): B^n \to B$ $B=\{0,1\}$ $x = \{x_1, x_2, \ldots x_n\}$

- $x_1, x_2, \ldots x_n$ are variables
- Each vertex of B^n is mapped to either 0 or 1
- The on-set of f is $\{x|f(x) = 1\} = f^1 = f^{-1}(1)$
- The off-set of f is $\{x|f(x) = 0\} = f^0 = f^{-1}(0)$
- If $f^1 = B^n$, f is a tautology
- If $f^0 = B^n$, i.e. $f = \phi$, f is not satisfiable
- If $f(x) = g(x)$ for all $x \varepsilon B^n$, then f and g are equivalent
- Also referred to as logic functions
- How many logic functions are there?

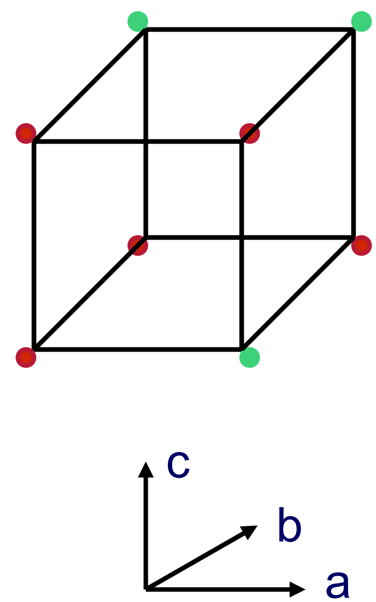


Representation of Boolean Functions

- The truth table for a function f: B^n ->B is a tabular representation of its value at each of the 2^n vertices of B^n.
- Example:

a b c	f
0 0 0	0
0 0 1	0
0 1 0	0
0 1 1	1
1 0 0	1
1 0 1	0
1 1 0	0
1 1 1	1

f = b c + a b' c'



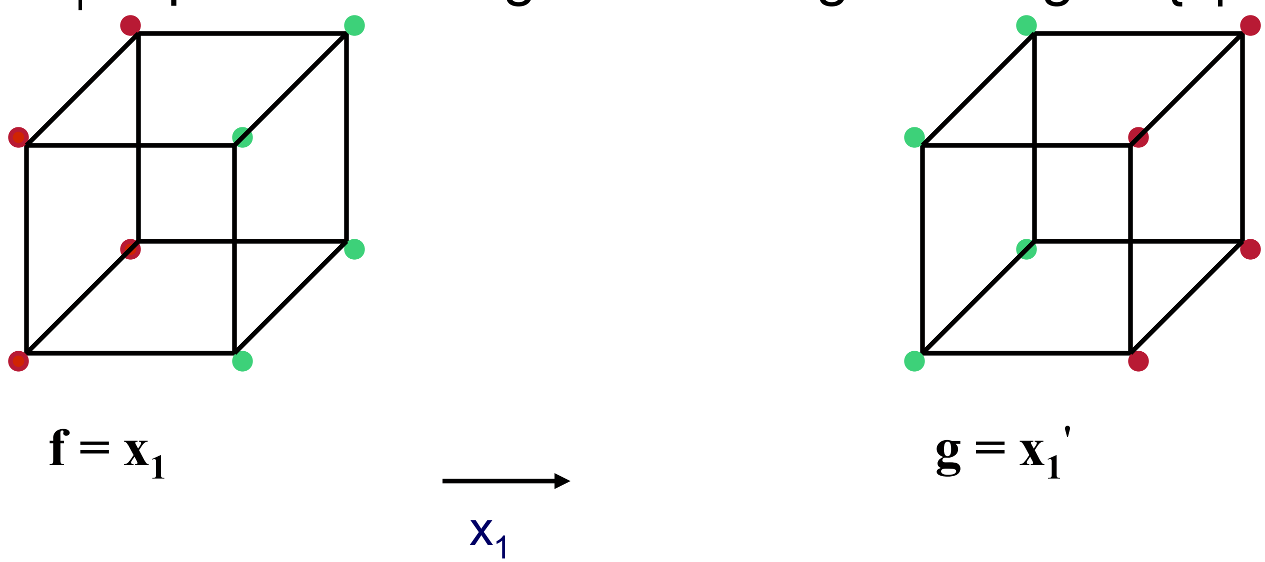
- Intractable for large n (but canonical).
- Canonical means that if two functions are equivalent, then their canonical representations are isomorphic.

Boolean Satisfiability

- Is there a any satisfying assignment for the function, i.e. is there at least one point in the ON-set of the function?
- How hard is this?
  - Depends on how the function is represented.
    - Boolean n-cube, truth table
      - Easy once we have the representation
      - But representation size is exponential in n ☹
    - How about other representation?
      - Boolean Formula
      - BDD
      - Circuit

Literals

- A literal is a variable or its negation.
  - x_1, x_1' (also represented as $\neg x_1$)
- Literal x_1 represents a logic function f where $f^1 = \{x|x_1=1\}$
- Literal x_1' represents a logic function g where $g^1 = \{x|x_1=0\}$



Boolean Formulas

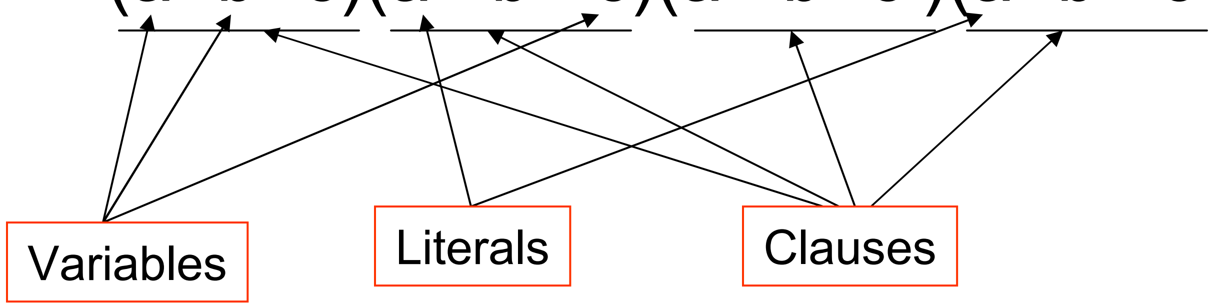
- Boolean functions can be represented as formulas defined as catenations of:
  - Parenthesis (,)
  - Literals x_1, x_1'
  - Boolean operators + (OR), x or . (AND), NOT
  - NOT (Negation) : $f' = h$ such that $h^1 = f^0$
  - AND (Conjunction): (f AND g) = h such that $h^1 = \{x | f(x) = 1 \text{ and } g(x) = 1\}$
  - OR (Disjunction) : (f OR g) = h such that $h^1 = \{x | f(x) = 1 \text{ or } g(x) = 1\}$
- Usually replace x with catenation
  - e.g. $x_1 \times x_2$ with $x_1 x_2$
- How many formulas can we have with n variables?
- Examples:
  - $f = x_1 x_2' + x_1' x_2$
    $= (x_1 + x_2)(x_1' + x_2')$
  - $h = x_1 + x_2 x_3$
    $= (x_1' (x_2' + x_3'))'$

Boolean Satisfiability (SAT)

- Given a Boolean propositional formula, determine whether there exists a variable assignment that makes the formula evaluate to *true*.

- Formulas are often expressed in *Conjunctive Normal Form (CNF)*

$$(a+b+c)(a'+b'+c)(a'+b+c')(a+b'+c')$$



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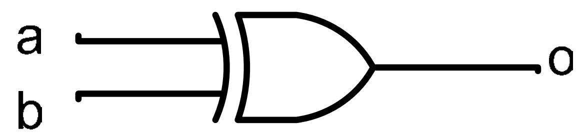
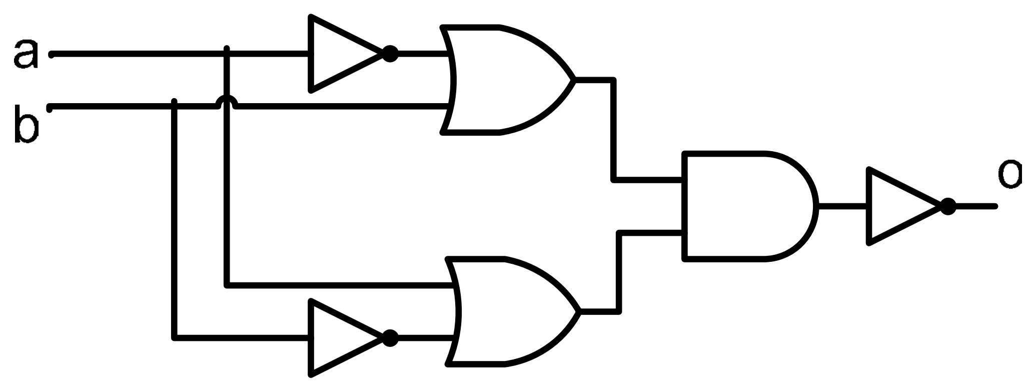
$$(a+b+c)(a'+b'+c)(a'+b+c')(a+b'+c')$$

$$(a+b)(a'+b)(a+b')(a'+b')$$

Convert a Boolean Circuit into CNF

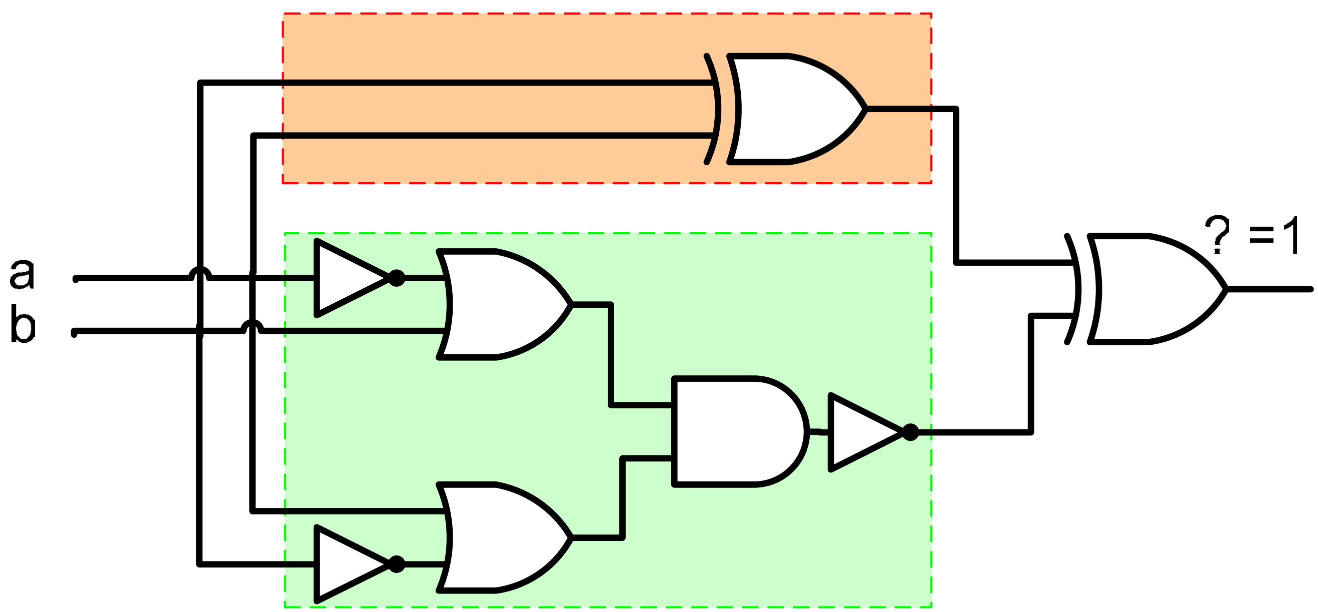
- Example: Combinational Equivalence Checking



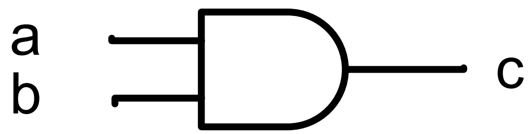


Combinational Equivalence Checking

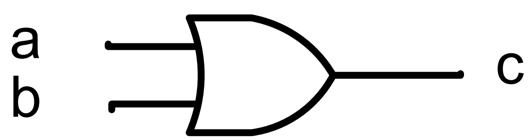
- Miter Circuit



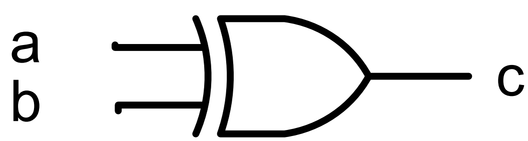
Modeling of Combinational Gates



(a + c')(b + c')(a' + b' + c)

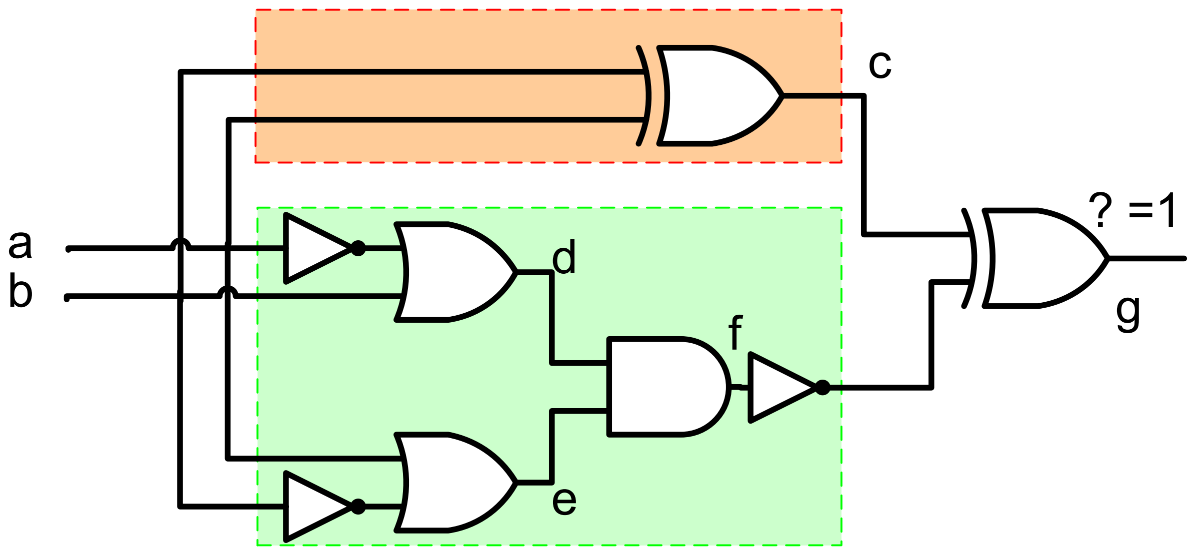


(a' + c)(b' + c)(a + b + c')



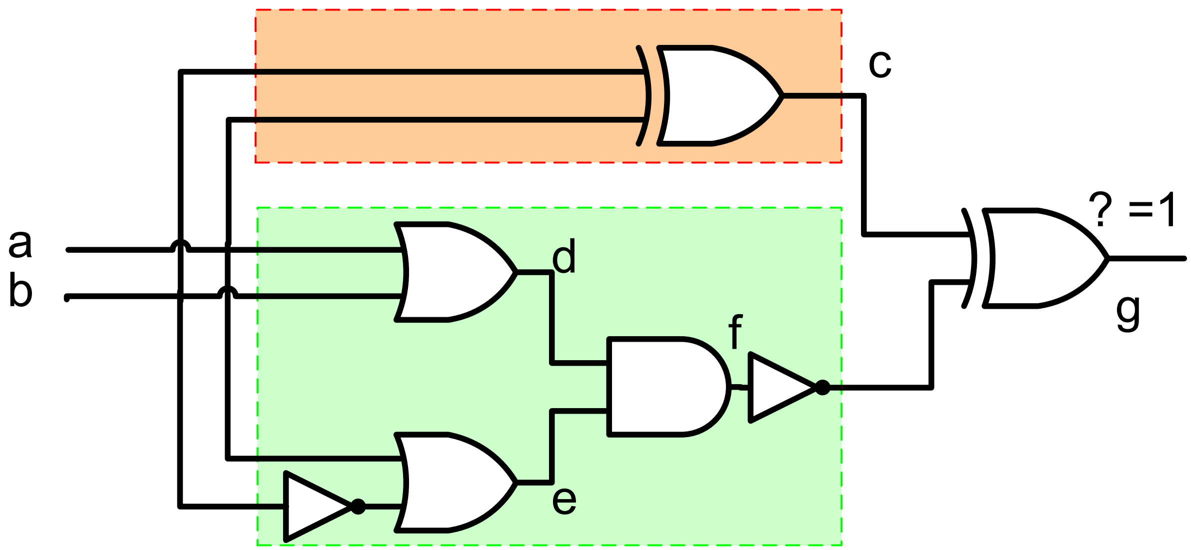
(a' + b' + c')(a + b + c')(a + b' + c)(a' + b + c)

From Combinational Equivalence Checking to SAT



(a’ + b’ + c’)(a + b + c’)(a + b’ + c)(a’ + b + c)
(a + d)(b’ + d)(a’ + b + d’)
(a’ + e)(b + e)(a + b’ + e’)
(d + f’)(e + f’)(d’ + e’ + f)
(c’ + f + g’)(c + f’ + g’)(c + f + g)(c’ + f’ + g)
(g)

From Combinational Equivalence Checking to SAT



(a' + b' + c')(a + b + c')(a + b' + c)(a' + b + c)

(a' + d)(b' + d)(a + b + d')

(a' + e)(b + e)(a + b' + e')

(d + f')(e + f')(d' + e' + f)

(c' + f + g')(c + f' + g')(c + f + g)(c' + f' + g)

(g)

Convert an Arbitrary Boolean Formula into CNF

- It is possible to convert an arbitrary function into CNF
  - Without introducing new variables, the size of the resulting formula will grow exponentially
    - Not practical
  - By introducing intermediate variables, the size of the resulting formula can grow linearly
    - How?
    - Number of intermediate variable equal to the number of Boolean operations
    - The resulting formula will have the same satisfiability as the original one
- It's sufficient for a SAT solver to solve problems in CNF
  - Almost all modern SAT solver operates on CNF

Complexity of SAT

- A CNF formula is said to belong to *k*-SAT if each clause of the formula contains no more than *k* literals.

- Classic Result:
  - Cook 1971: 3-SAT problem is NP-Complete.
  - NP complete: Class of problems for which no known solutions exists that takes less than $O(2^n)$ steps. However, it has not been proved that the problem needs at least an exponential number of steps. The common conjecture is that it does.
  - k-SAT is NP-complete for $k \geq 3$.

- The obvious lower bound for a SAT problem with n variables is 2^n.

- Currently, the best lower bound for a SAT problem with n variables is due to Paturi etc., E.g. for satisfiable 3-SAT, the complexity for finding a solution is $O(2^{0.448n})$.

SAT Problems with Polynomial Complexity

- Some special SAT classes can be solved in polynomial time.
  - If a problem is solvable in polynomial time, we can use special algorithms to solve them efficiently.
  - Part of the original problem may belong to a polynomial solvable class, it is possible to exploit this property during the solving process. (e.g. Larrabee).
  - During the solution process, a problem state may evolve to one that has a polynomial solution. We can exploit heuristics that are likely to reduce a problem to one that is solvable in polynomial time quickly (e.g. SATO).
- 2-SAT problems can be solved in linear time wrt the size of the problem (Aspvall, Plass and Tarjan, 1979).
- A Horn formula can be solved in linear time wrt the size of the formula.

Horn Formulas

- Horn sentences are often generated from knowledge base reasoning:
  - rules: if x, y, z are true, then r is true
  - $xyz \rightarrow r$
  - $a \rightarrow b$
    - If a is true, then b must be true to make the formula true
    - if a is false, then the formula is true
    - $(a' + b)$
  - $xyz \rightarrow r$: $(x' + y' + z' + r)$

- A CNF formula is Horn if every clause has at most one positive literal
  - What does it mean if a clause contains no positive literal?
  - What does it mean if a clause contains only one positive literal and no negative literal?

- A Horn formula can be solved in linear time wrt the size of the formula.
  - Do unit implication until no unit clause exists
  - If conflict, the formula is unsatisfiable
  - Else the formula can be satisfied by assigning all the unassigned variables with value 0

Problem Hardness and Phase Transition

- Not all SAT problems are hard
  - Many practical SAT instances can be solved very efficiently
  - The theory of NP-completeness is based on *worst-case* complexity.
  - To explain the behavior of algorithms in practice, the theory of *average-case* complexity is more appropriate.

- Use random generated SAT instances to explore the hardness distribution
  - Very different characteristics from the instances generated from real world applications
  - But are of great theoretical interests

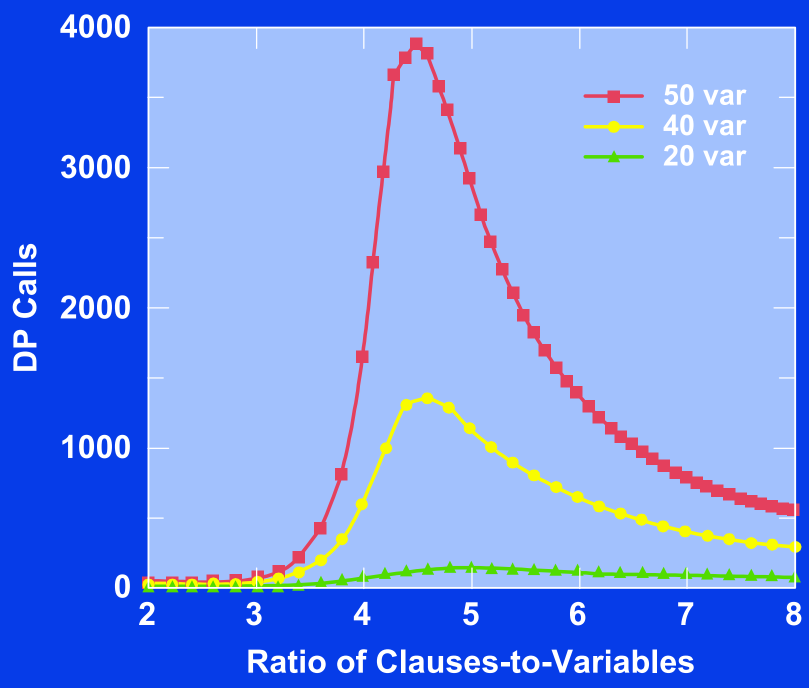
Fixed-clause length model

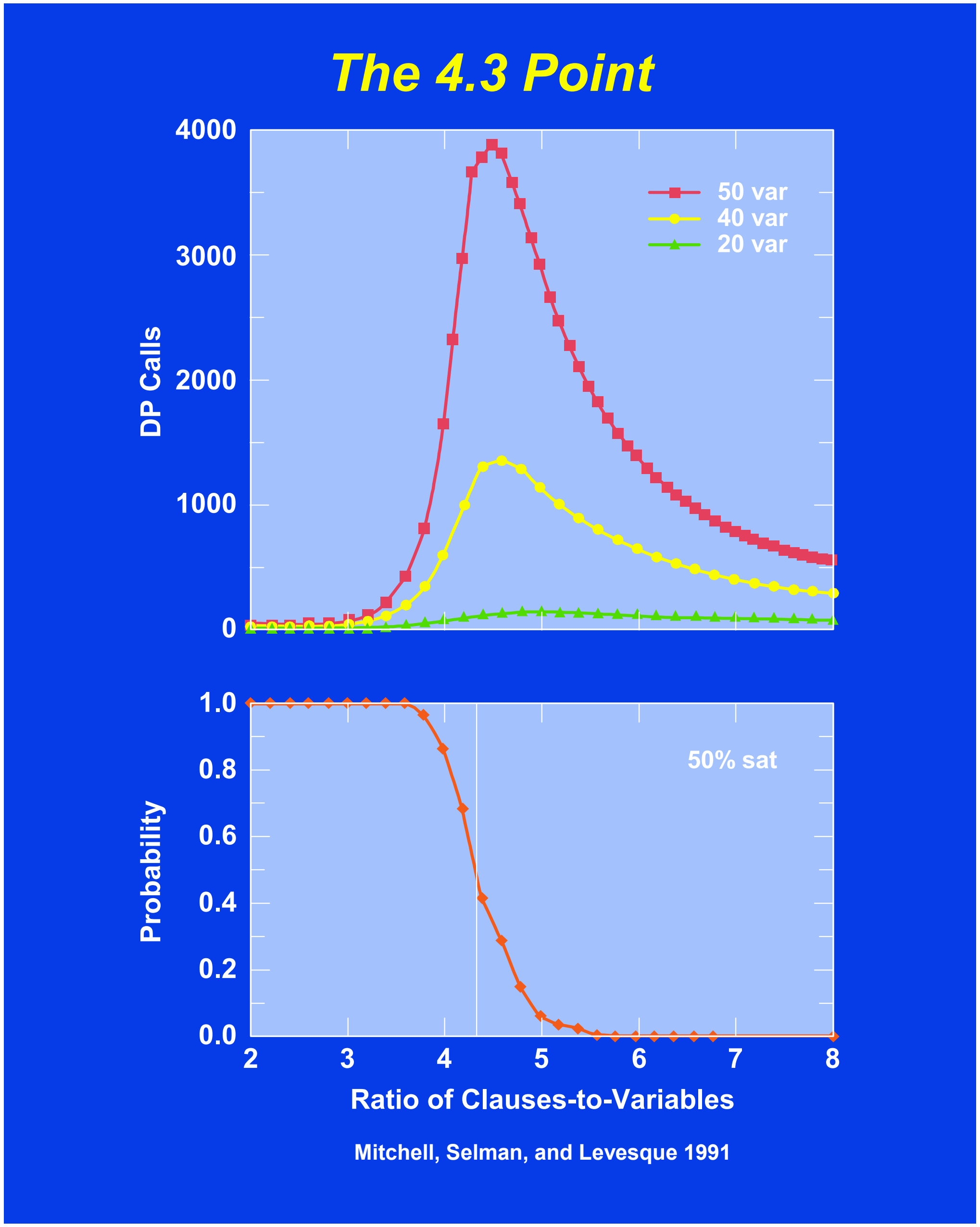
- Generated by selecting clauses uniformly at random from the set of all possible (non-trivial) clauses of a given length, random k-SAT.
- Three parameters: the number of variables N, the number of literals per clause K, and the number of clauses L.
  - Formulas with few clauses: under-constrained (usually satisfiable),
  - Formulas with many clauses: over-constrained (usually unsatisfiable)
  - Both under-constrained and over-constrained problems are much easier than problems of medium length

Phase transition behavior

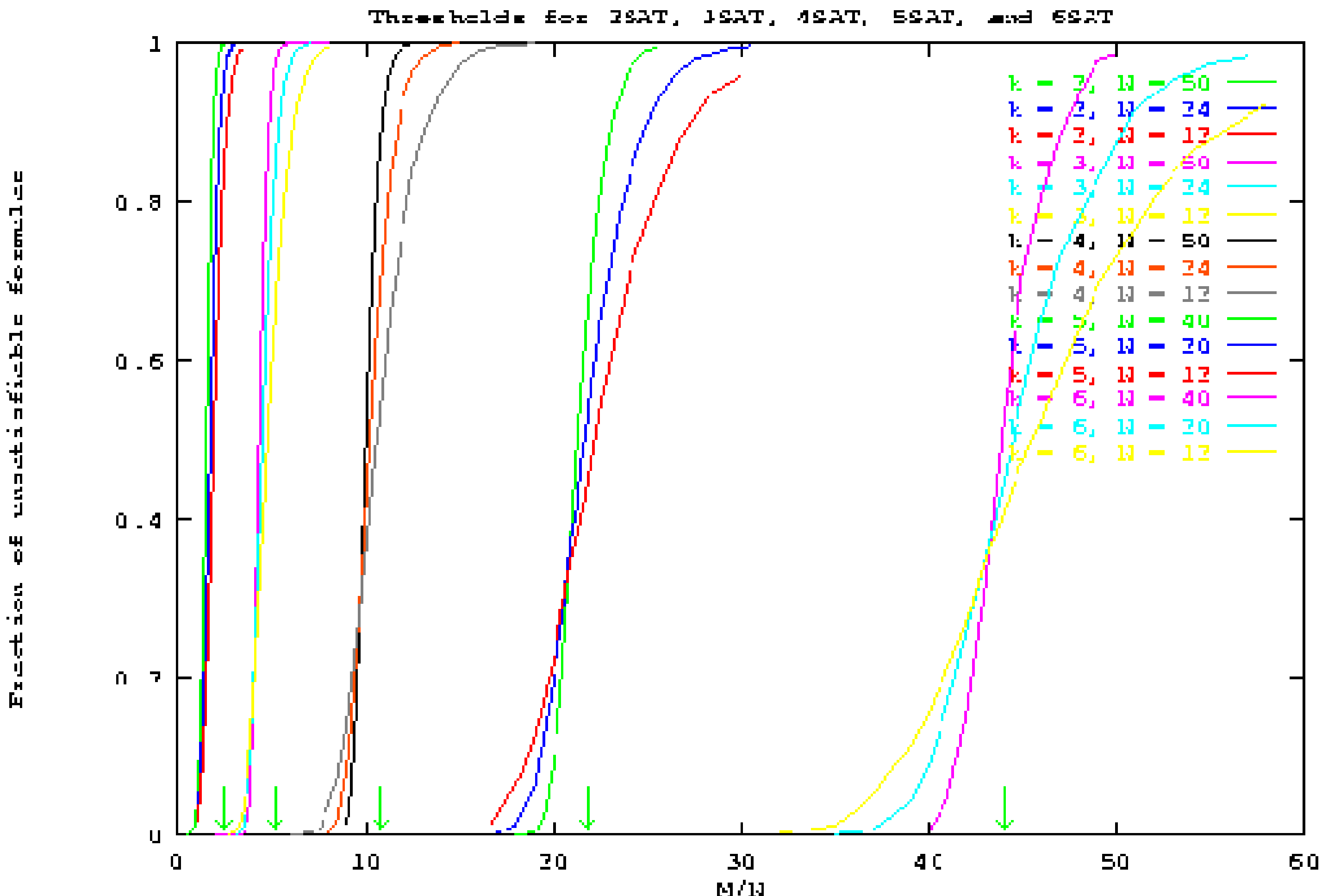
- Problems which are very over-constrained are unsatisfiable and it is usually easy to determine this. Problems which are very under-constrained are satisfiable and it is usually easy to guess one of the many solutions.
- A phase transition tends to occur in between when problems are critically constrained, and it is difficult to determine if they are satisfiable or not.
- For random 2-SAT, the phase transition has been proven to occur at L/N=1.
- For random 3-SAT, the phase transition has been experimentally show to occur around L/N = 4.3

Hardness of 3SAT




The 4.3 Point
4000
3000
2000
1000
0
DP Calls
50 var
40 var
20 var
1.0
0.8
0.6
0.4
0.2
0.0
Probability
50% sat
2
3
4
5
6
7
8
Ratio of Clauses-to-Variables
Mitchell, Selman, and Levesque 1991

Phase transition 2-, 3-, 4-, 5-, and 6-SAT



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Research

Threshold phenomena

- Threshold conjecture: for each k, there is some c* such that for each fixed value of c<c*, random k-SAT with n variables and cn clauses is satisfiable with probability tending to 1 as $n \to \infty$, and when c>c*, unsatisfiable with probability tending to 1.
- For the case of random 2-SAT, the conjecture has been shown true, and c*=1.
- Current status:
  - 3SAT threshold lies between 3.42 ~ 4.51

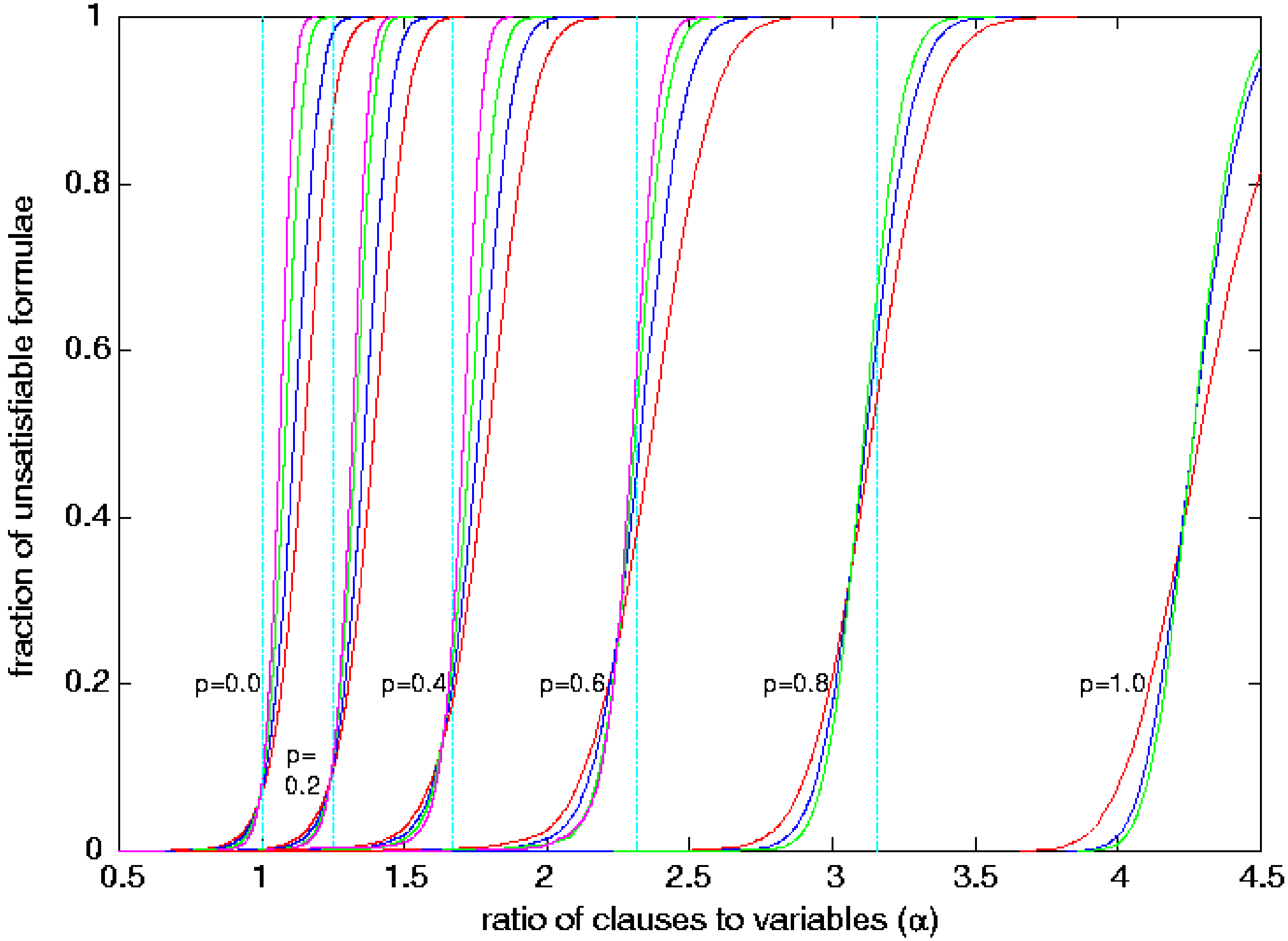
The 2+p-SAT model

- Mixtures of problem classes, e.g., 2-SAT and 3-SAT ("moving between P and NP")
- Mixture of binary and ternary clauses

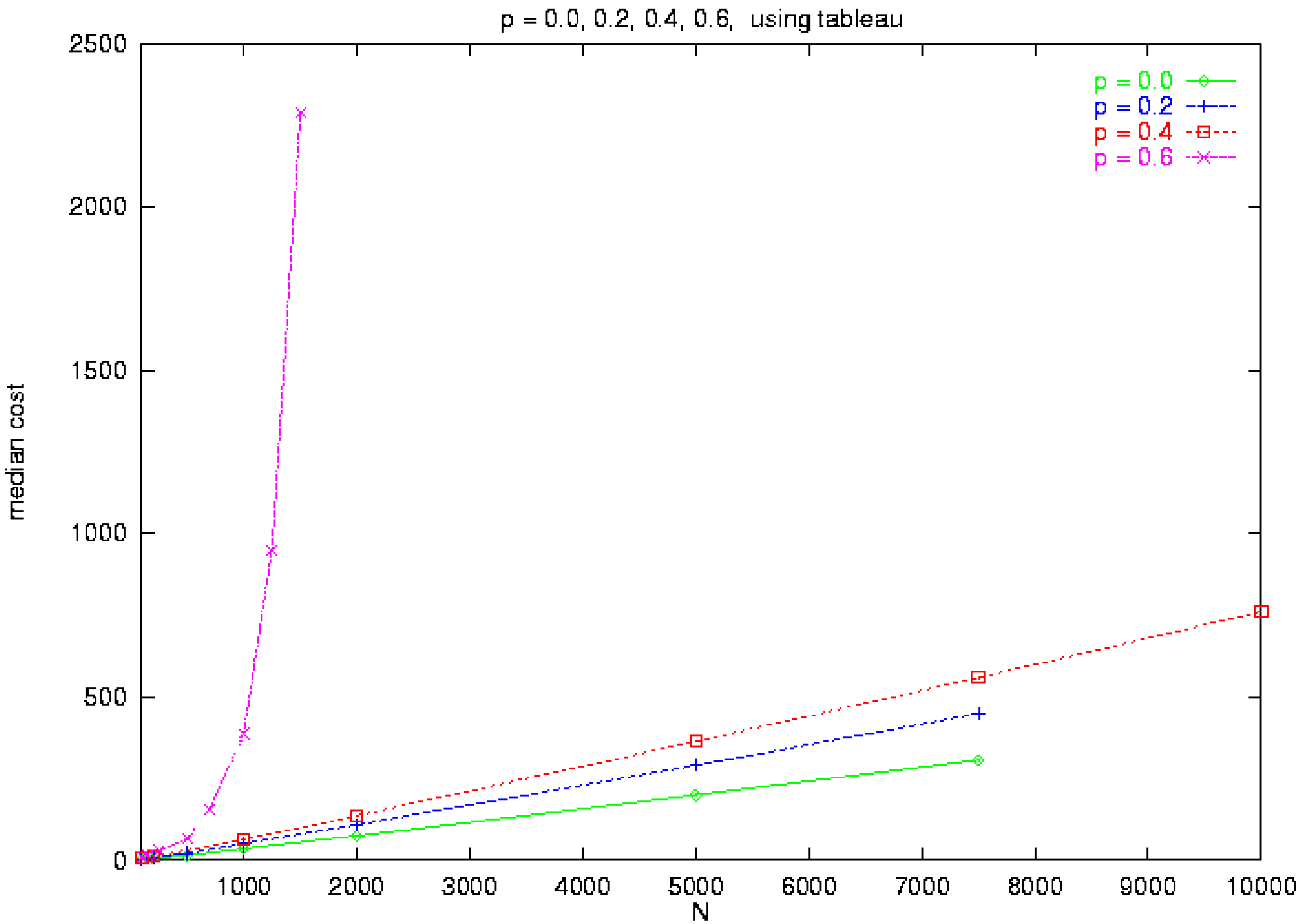
    p = fraction ternary

    p = 0.0 --- 2-SAT / p = 1.0 --- 3-SAT

Phase Transition for 2+p-SAT



Computational Cost



2+P Model

- p < ~ 0.41 --- model essentially behaves as 2-SAT
  - search proc. “sees” only binary constraints
  - smooth, continuous phase transition
- p > ~ 0.41 --- behaves as 3-SAT (exponential scaling)
  - abrupt, discontinuous scaling

SAT Algorithm: An Overview

- Davis, Putnam, 1960
  - Explicit resolution based
  - May explode in memory
- Davis, Logemann, Loveland, 1962
  - Search based.
  - Most successful, basis for almost all modern SAT solvers
  - Learning and non-chronological backtracking, 1996
- Stålmarcks algorithm, 1980s
  - Proprietary algorithm. Patented.
  - Commercial versions available
- Stochastic Methods, 1992
  - Unable to prove unsatisfiability, but may find solutions for a satisfying problem quickly.
  - Local search and hill climbing

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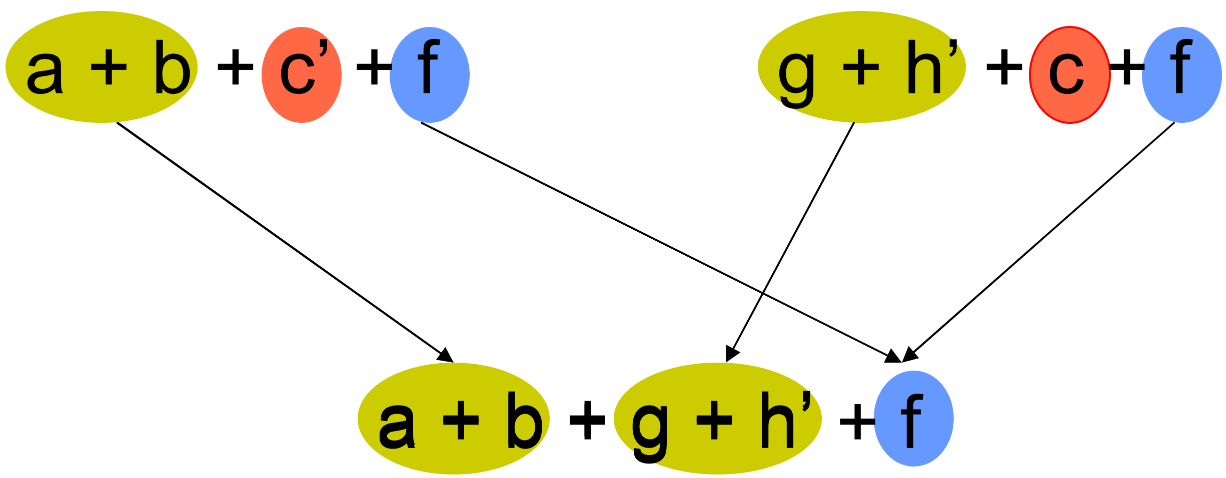
Resolution

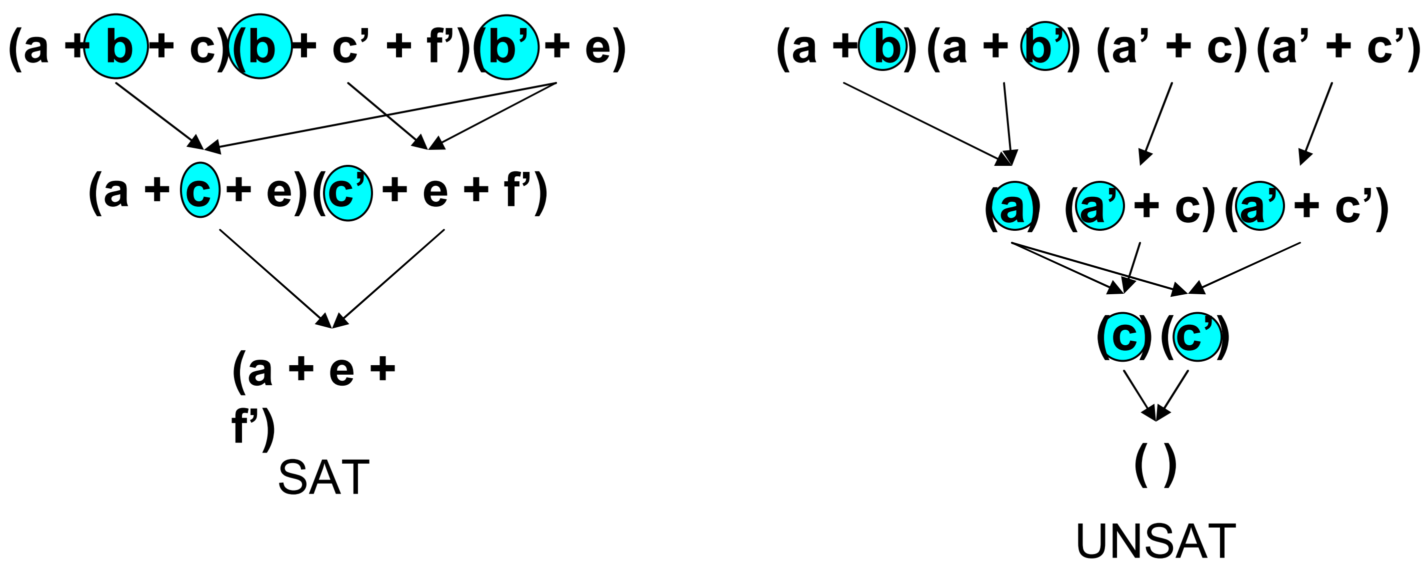
- Resolution of a pair of clauses with exactly **ONE** incompatible variable
  - Two clauses are said to have distance 1
  - (a+b)(a’+c) = (a+b)(a’+c)(b+c)



Davis Putnam Algorithm

M .Davis, H. Putnam, “A computing procedure for quantification theory", *J. of ACM*, Vol. 7, pp. 201-214, 1960

- Iteratively select a variable for resolution till no more variables are left.
- Can discard all original clauses after each iteration.



Potential memory explosion problem!

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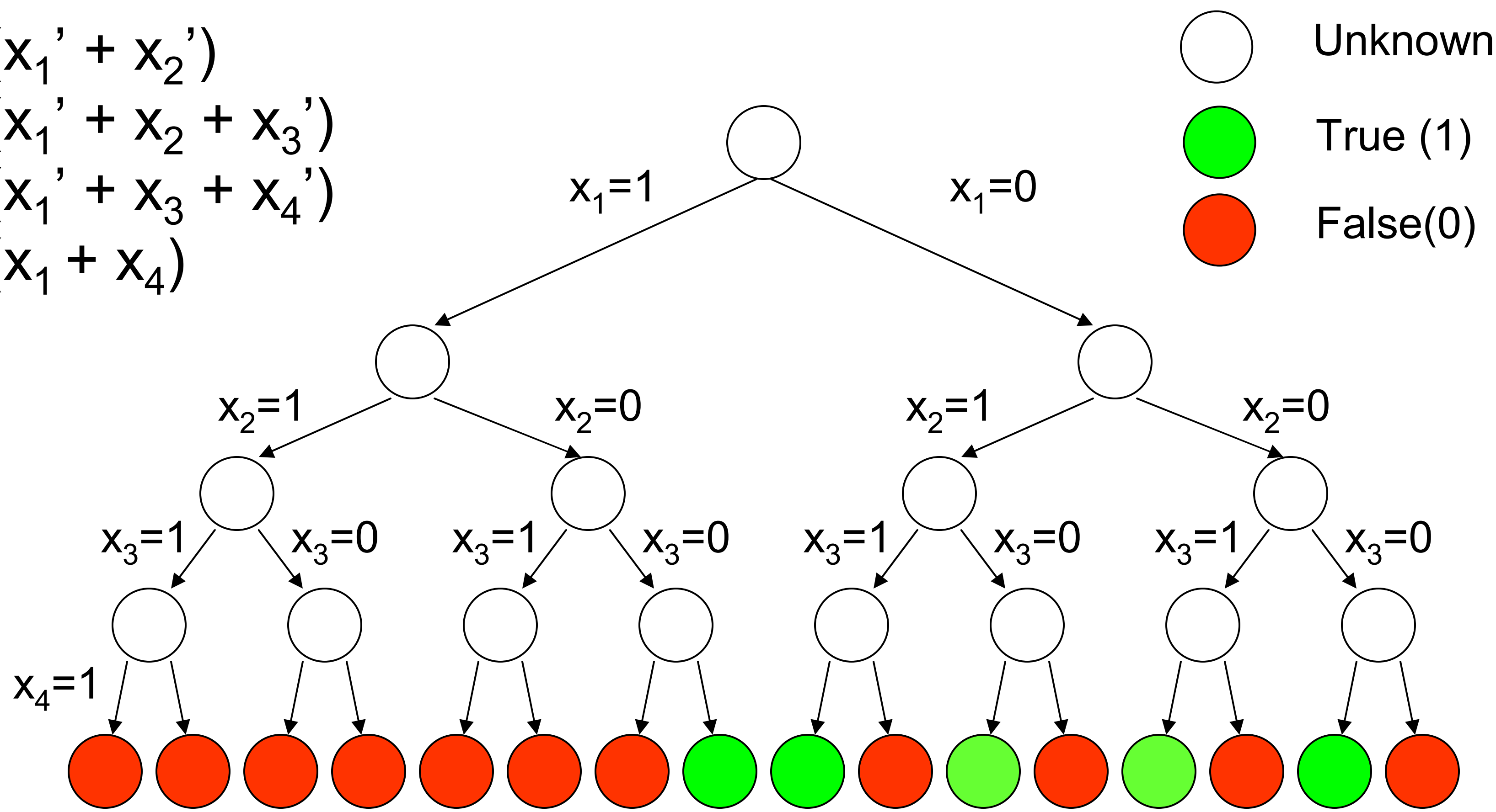
Search Tree of SAT Problem

$(x_1' + x_2')$
$(x_1' + x_2 + x_3')$
$(x_1' + x_3 + x_4')$
$(x_1 + x_4)$



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Deduction Rules for SAT

- Unit Literal Rule: If an unsatisfied clause has all but one of its literals evaluate to 0, then the *free* literal must be implied to be 1.

$$(a + b + c)(d' + e)(a + b + c' + d)$$

- Conflicting Rule: If all literals in a clause evaluate to 0, then the formula is unsatisfiable in this branch.

$$(a + b + c)(d' + e)(a + b + c' + d)$$

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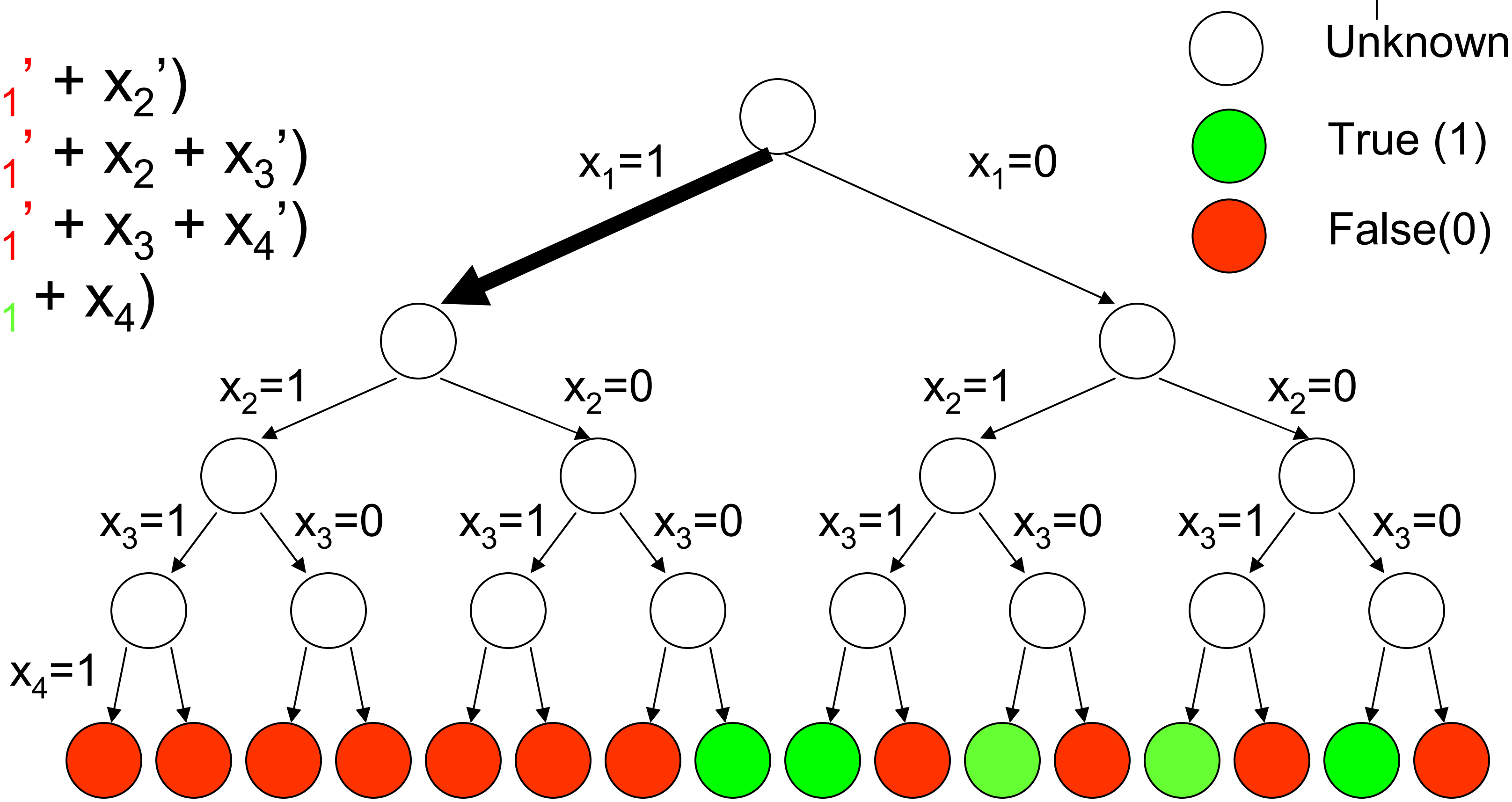

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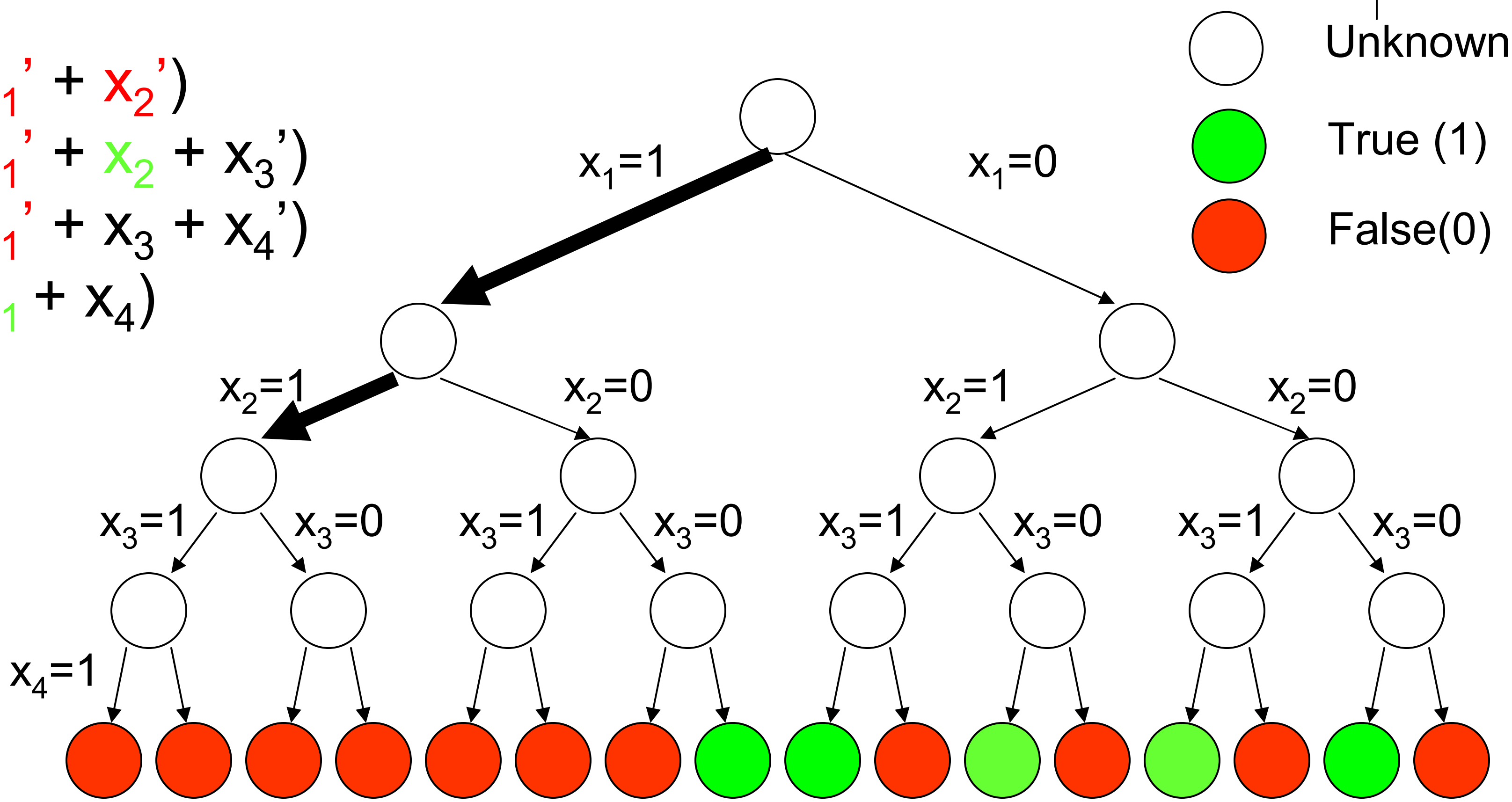

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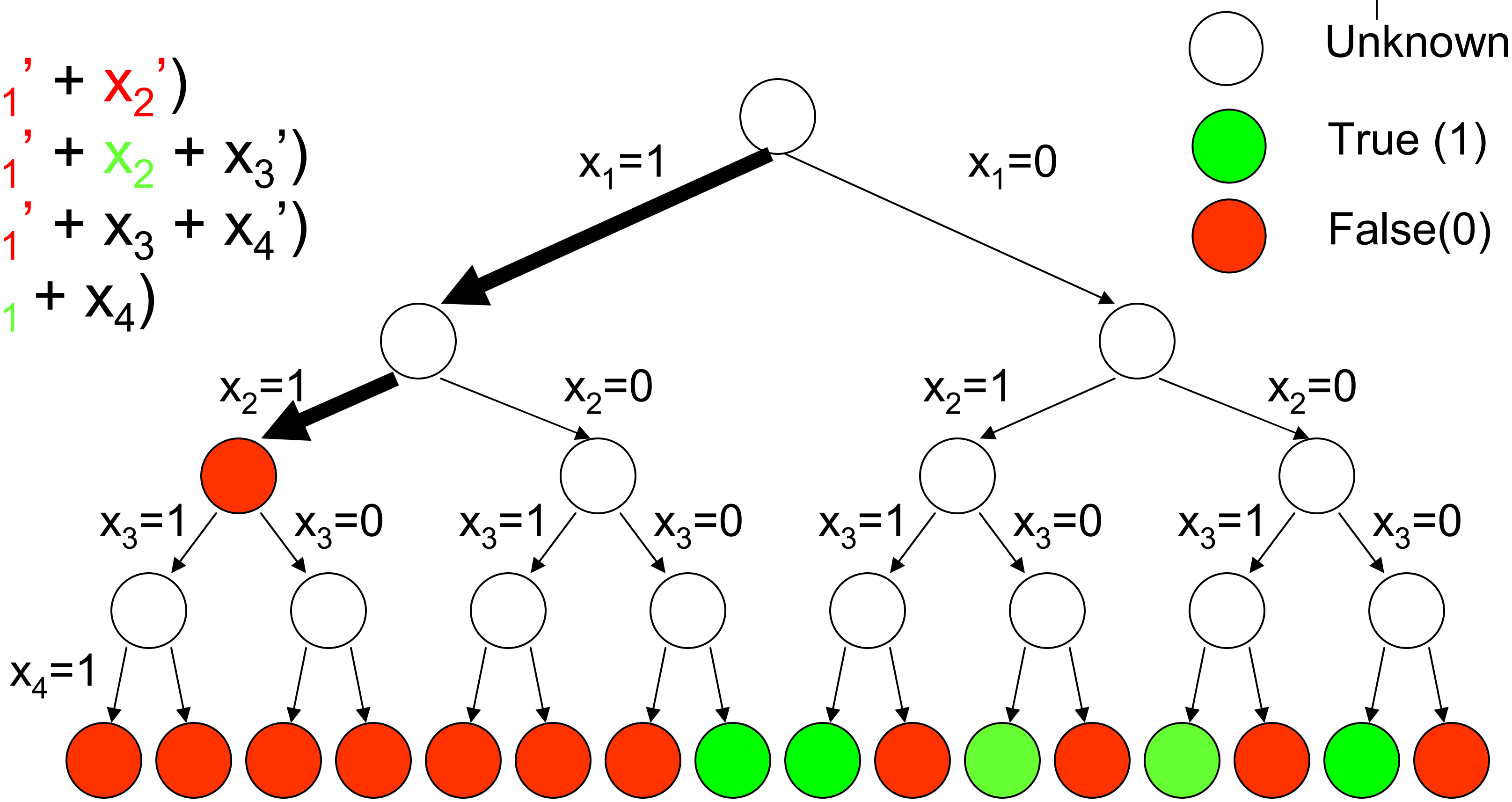

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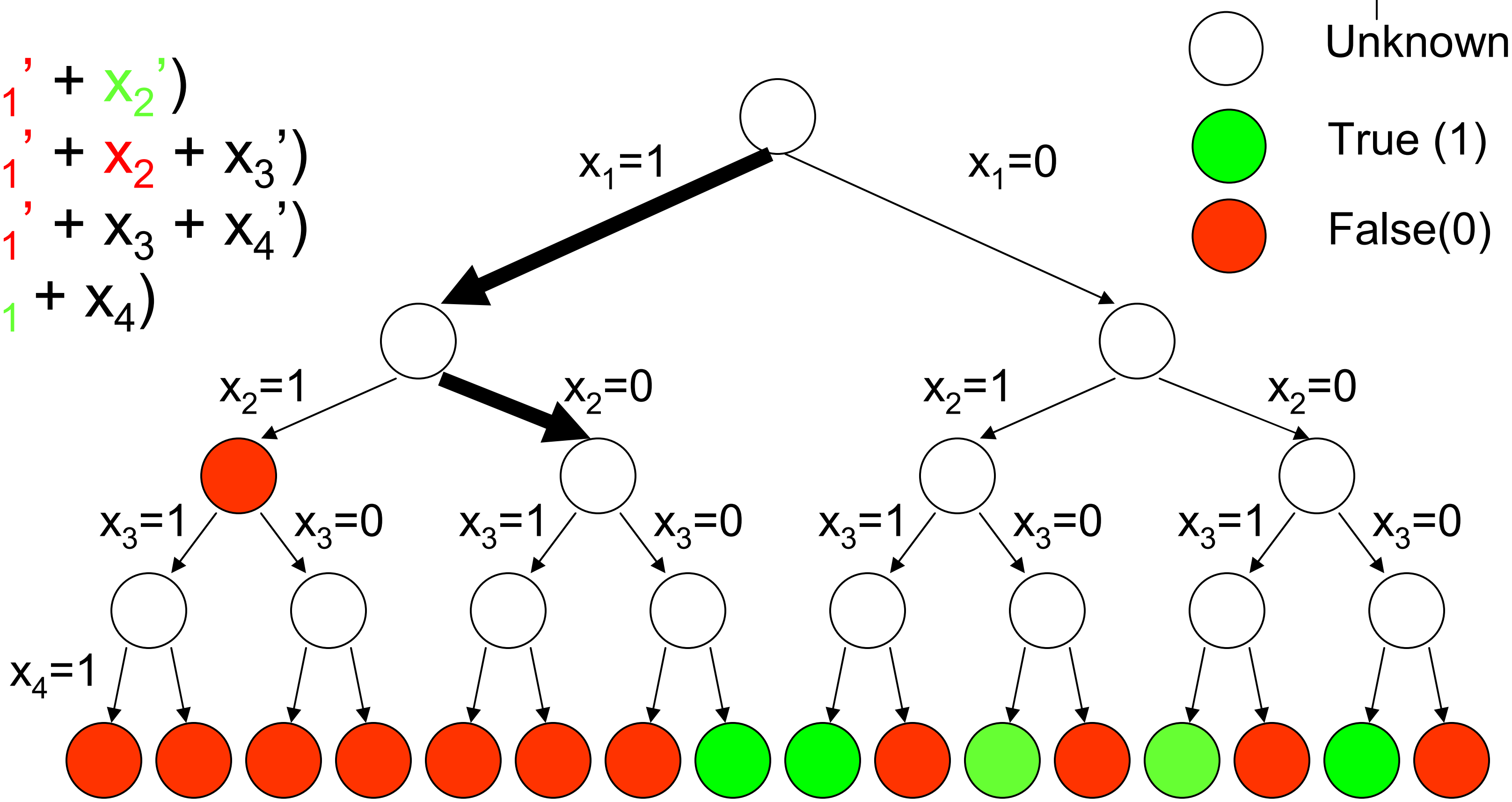

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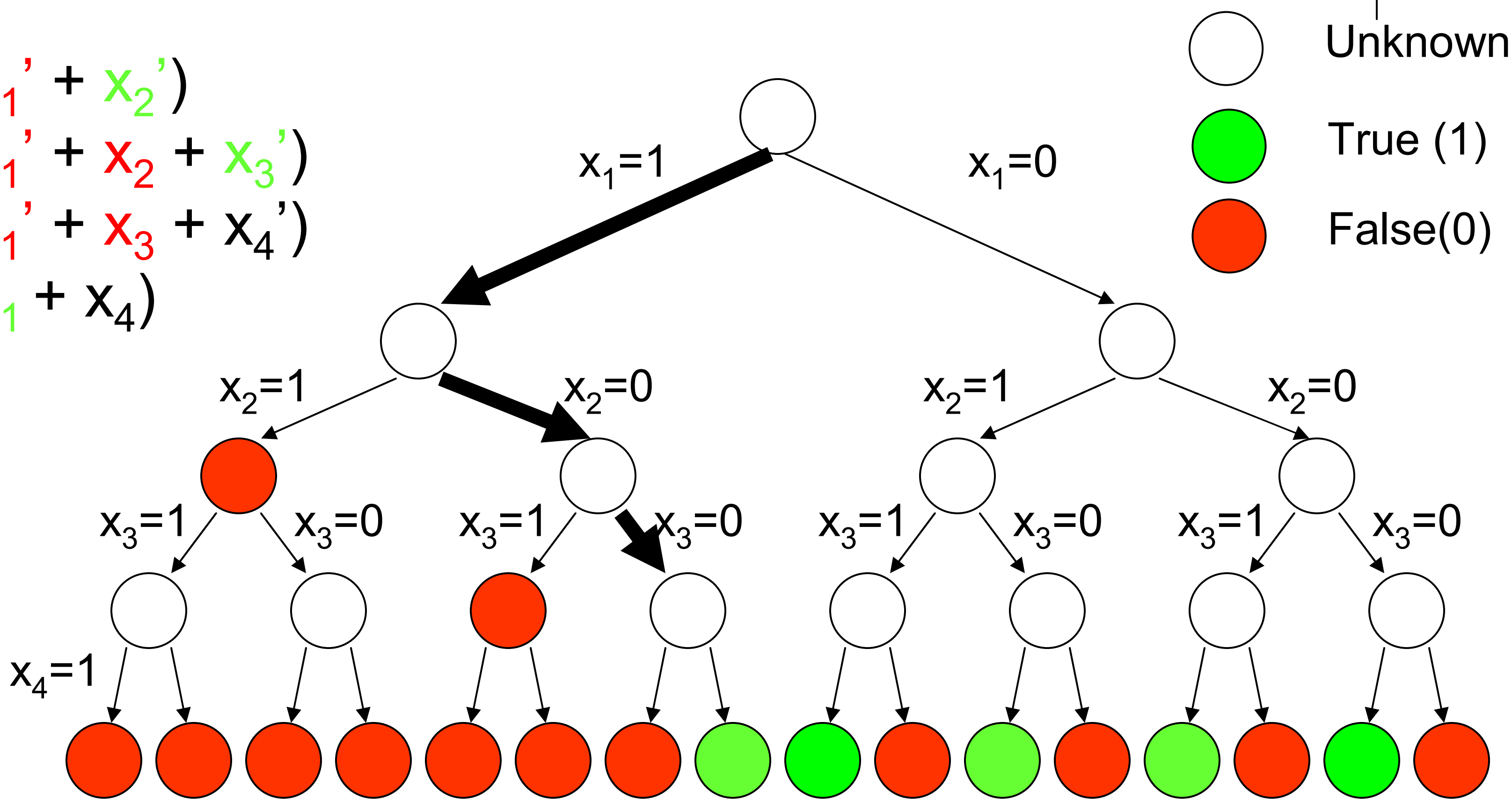


DLL Algorithm

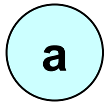
M. Davis, G. Logemann and D. Loveland, “A Machine Program for Theorem-Proving", *Communications of ACM*, Vol. 5, No. 7, pp. 394-397, 1962

- Basic framework for many modern SAT solvers
- Also known as DPLL for historical reasons

Basic DLL Procedure - DFS

(a' + b + c)
(a + c + d)
(a + c + d')
(a + c' + d)
(a + c' + d')
(b' + c' + d)
(a' + b + c')
(a' + b' + c)

Basic DLL Procedure - DFS



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(a' + b + c')
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Basic DLL Procedure - DFS

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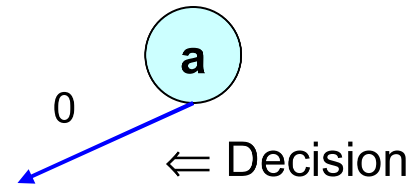
(a' + b' + c)



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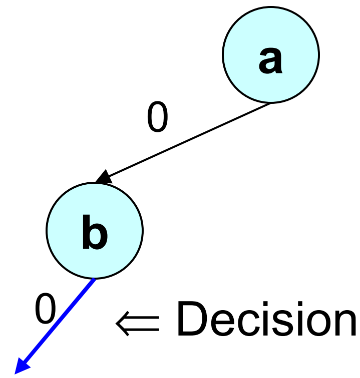

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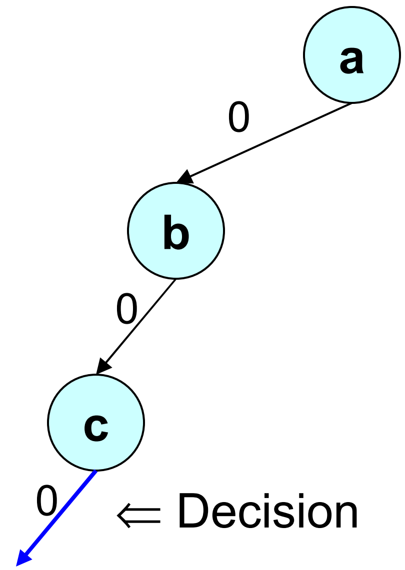

Microsoft Research

Basic DLL Procedure - DFS

(a' + b + c)
(a + c + d)
(a + c + d')
(a + c' + d)
(a + c' + d')
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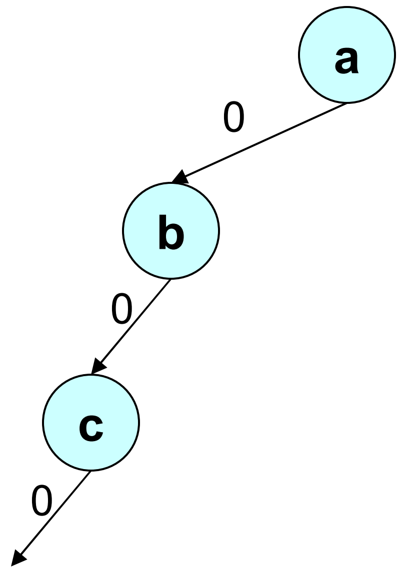
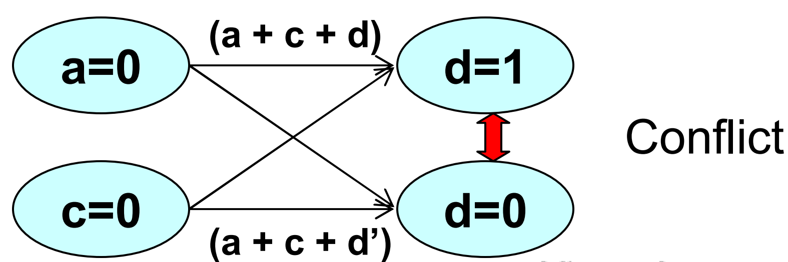
(a' + b' + c)



Implication Graph



Basic DLL Procedure - DFS

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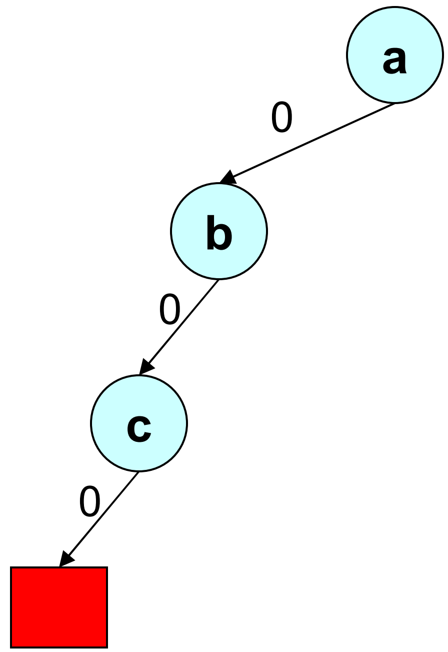
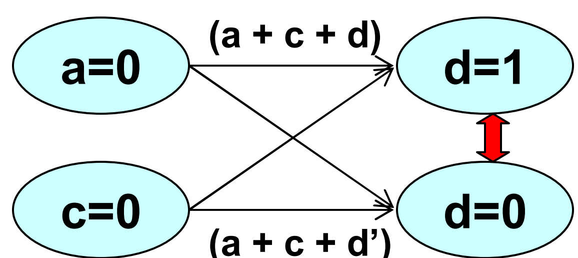


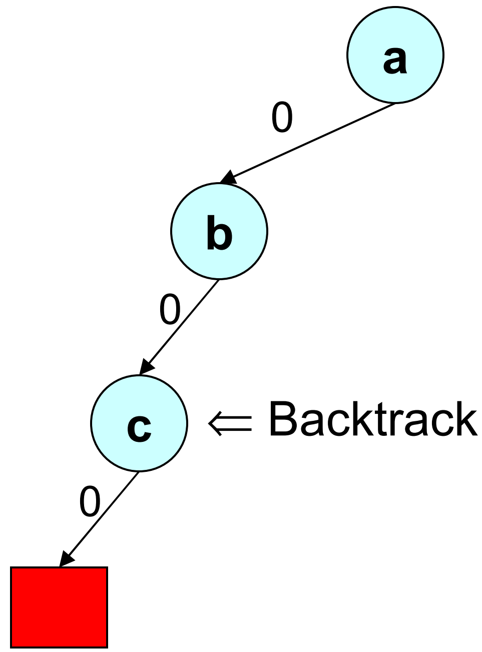
Implication Graph



Conflict!

Basic DLL Procedure - DFS

(a' + b + c)
(a + c + d)
(a + c + d')
(a + c' + d)
(a + c' + d')
(b' + c' + d)
(a' + b + c')
(a' + b' + c)



Basic DLL Procedure - DFS

(a’ + b + c)
(a + c + d)
(a + c + d’)
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(a + c’ + d’)
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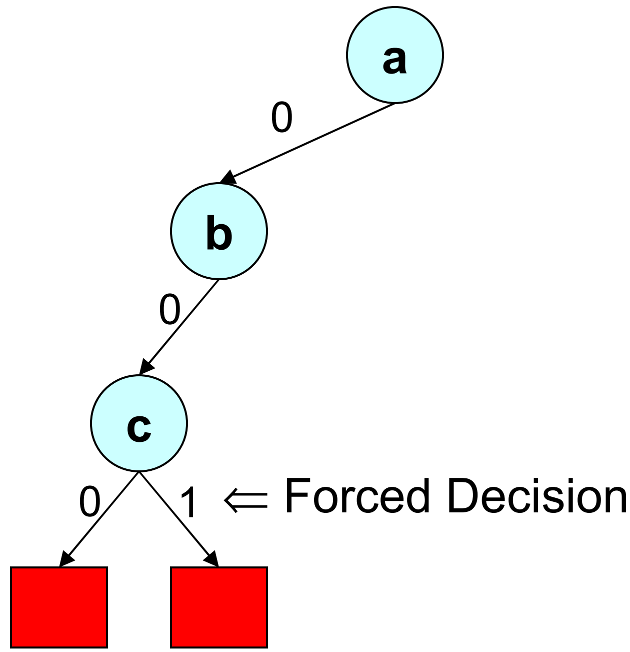

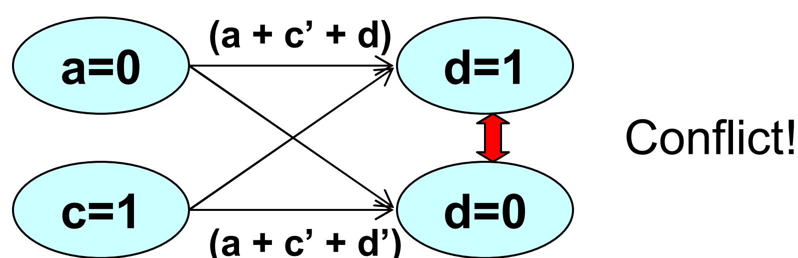

Basic DLL Procedure - DFS

(a' + b + c)
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(b' + c' + d)
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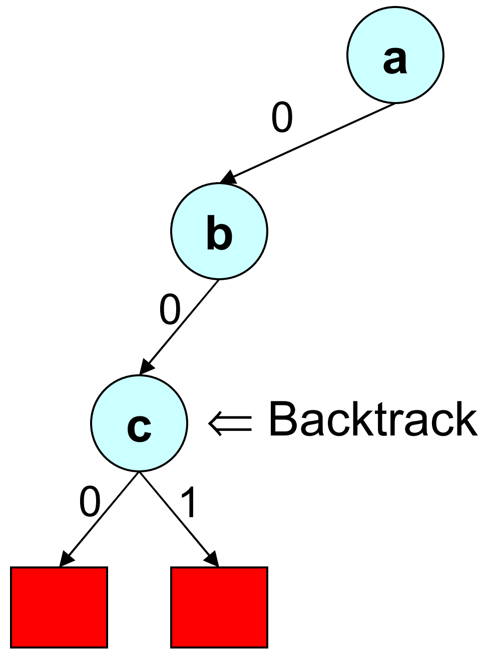

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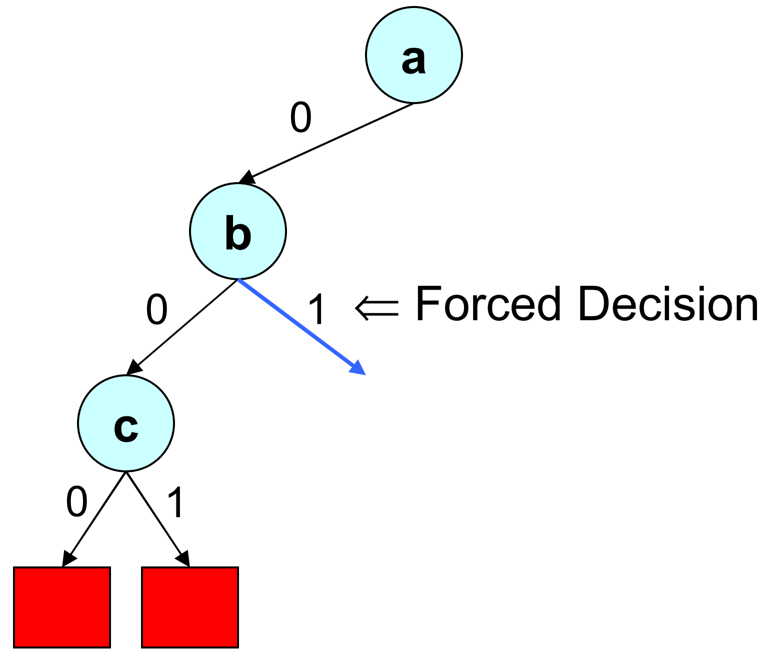

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(a + c + d)
(a + c + d')
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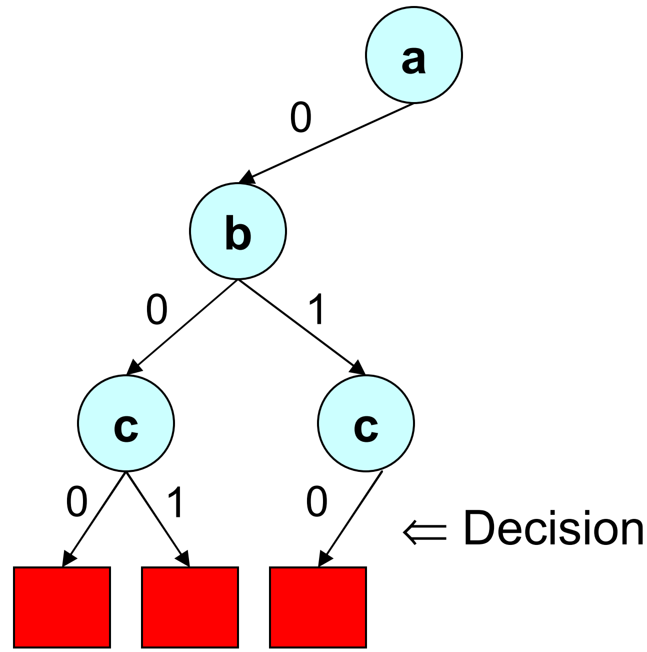

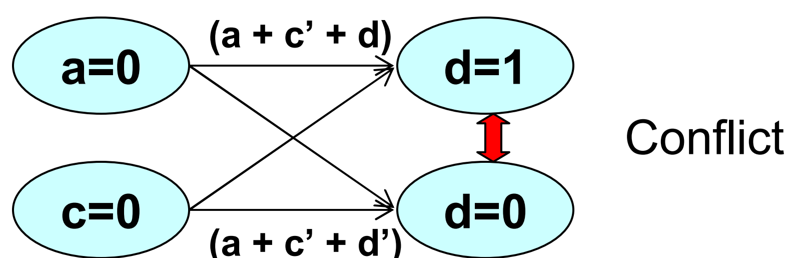

Basic DLL Procedure - DFS

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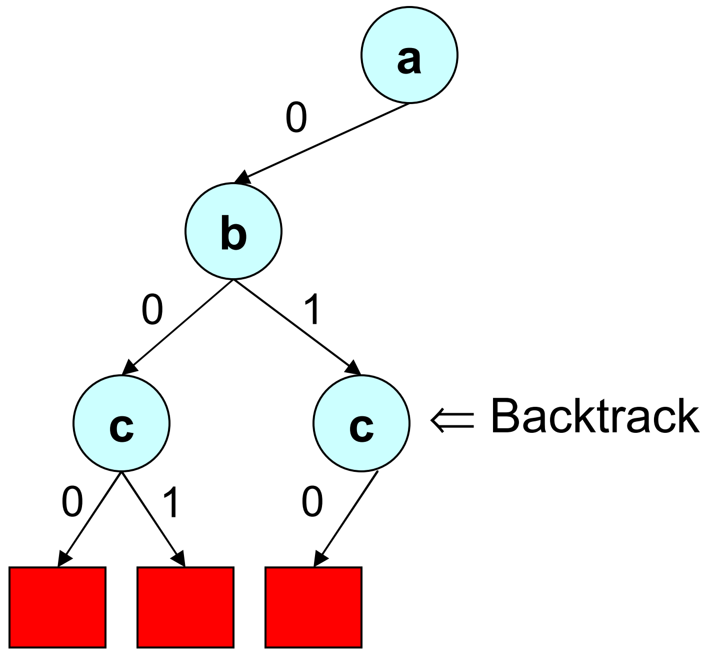

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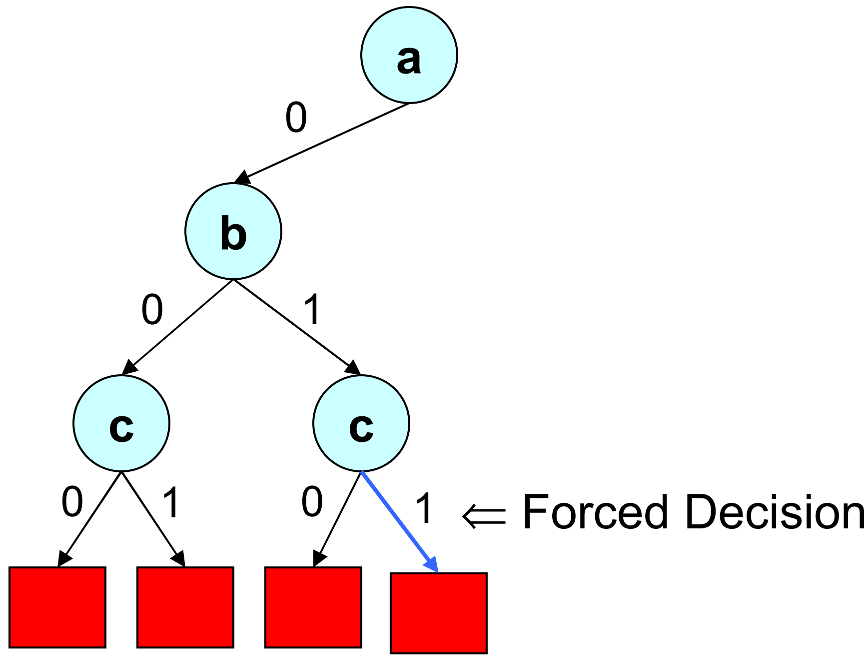

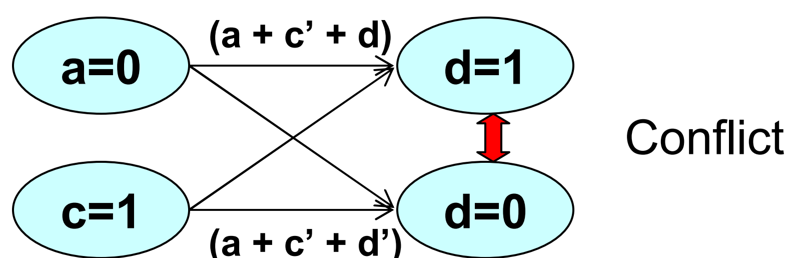

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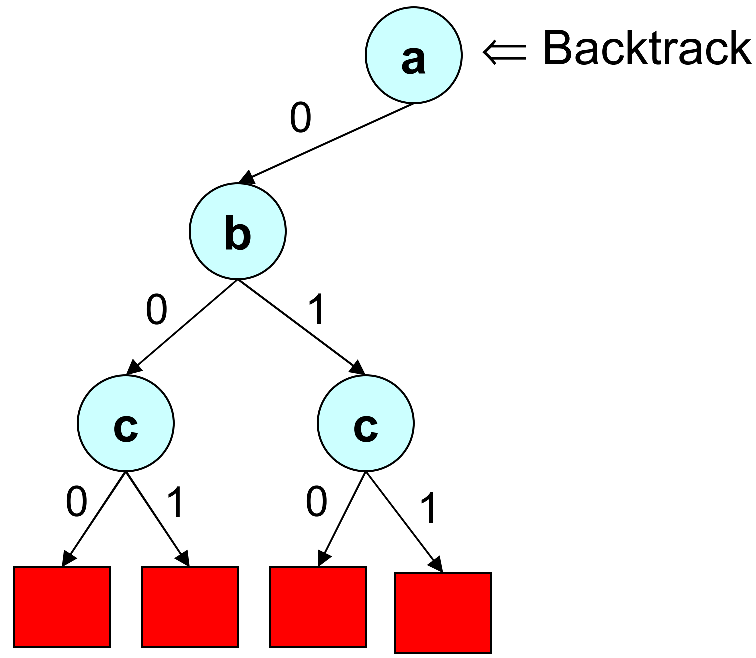

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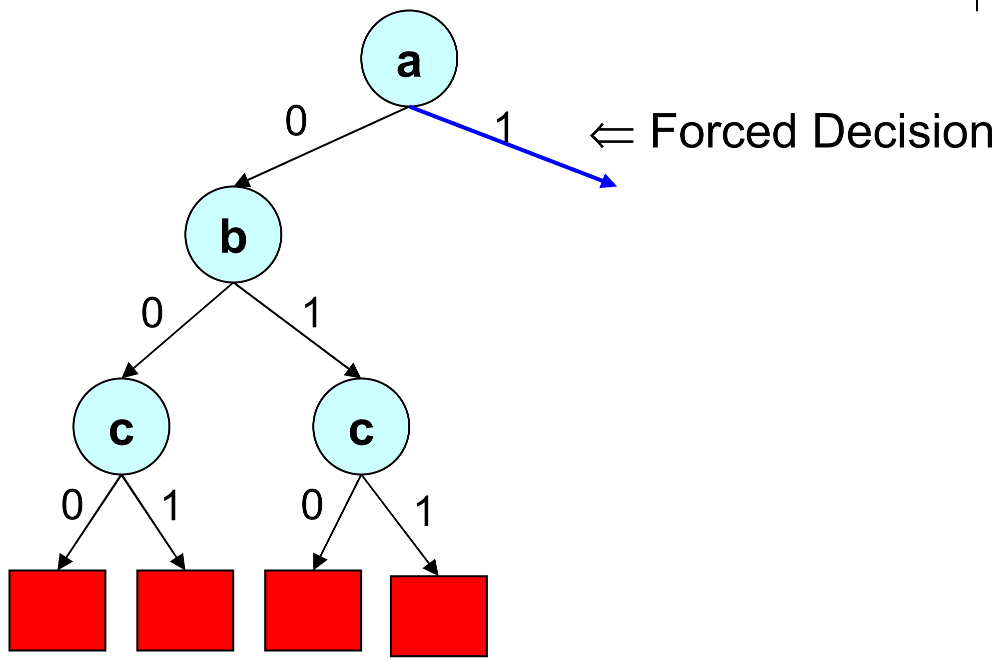

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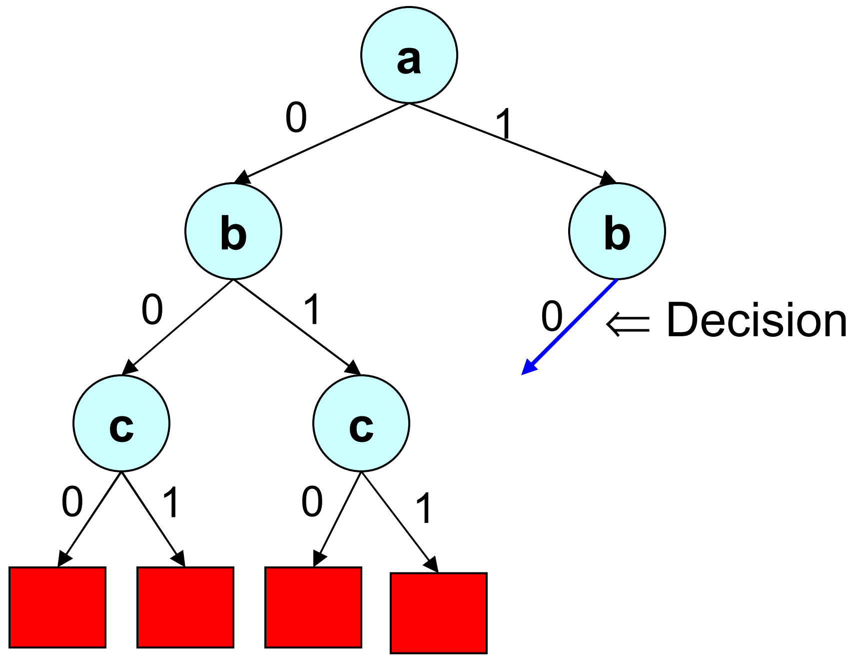

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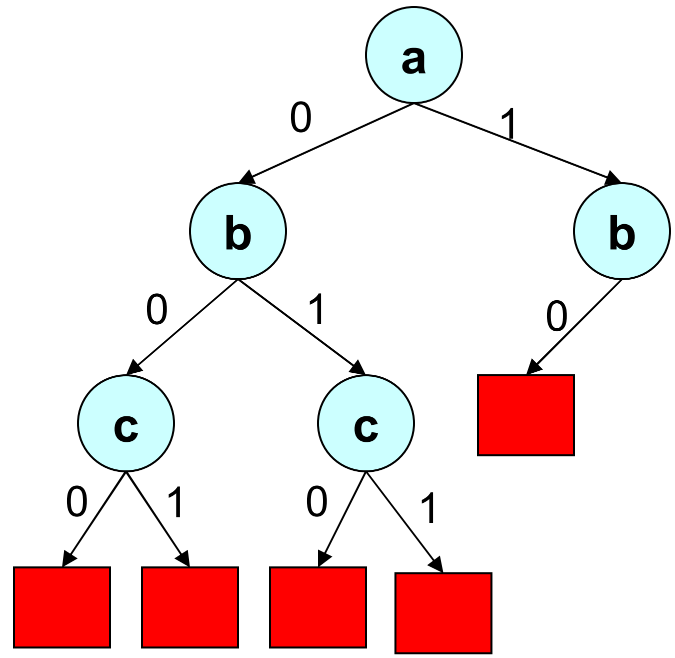

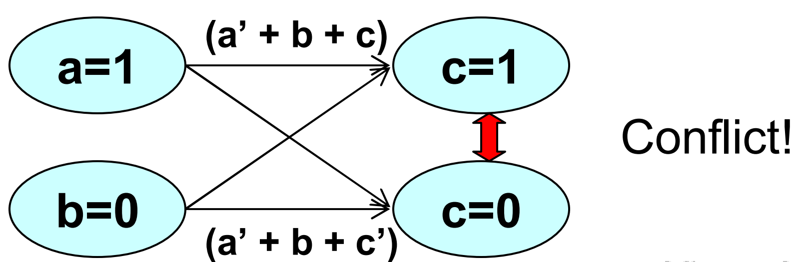

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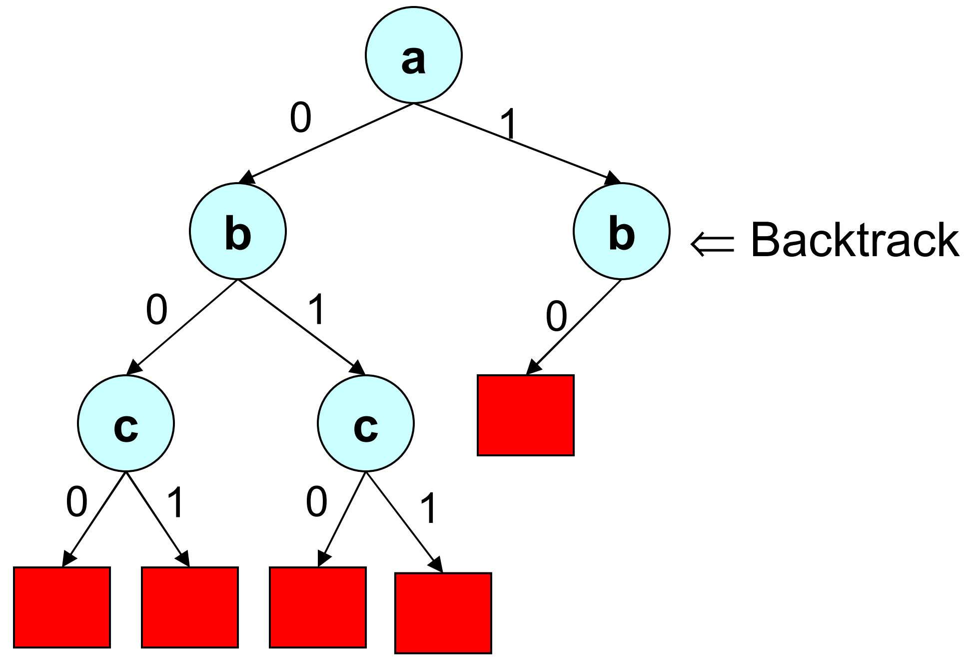

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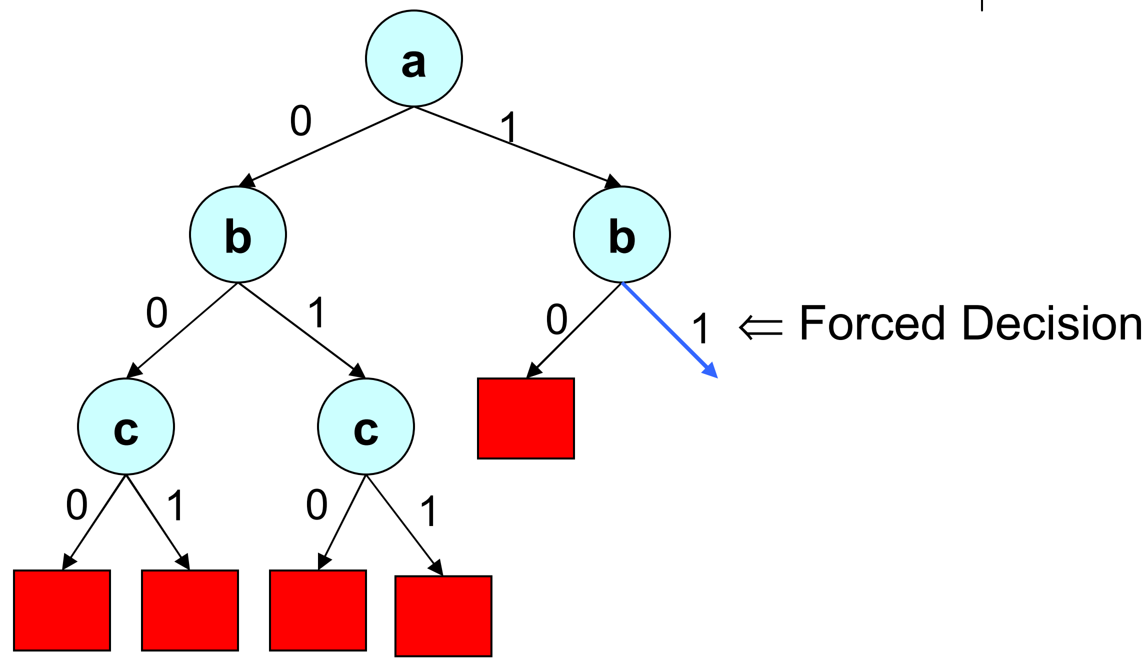

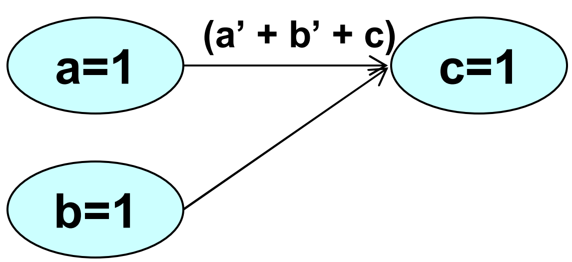

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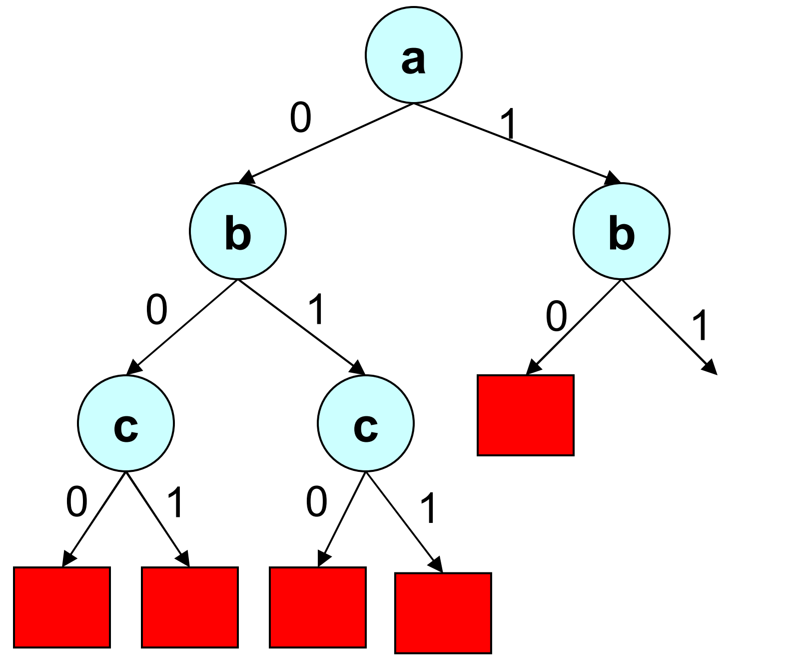

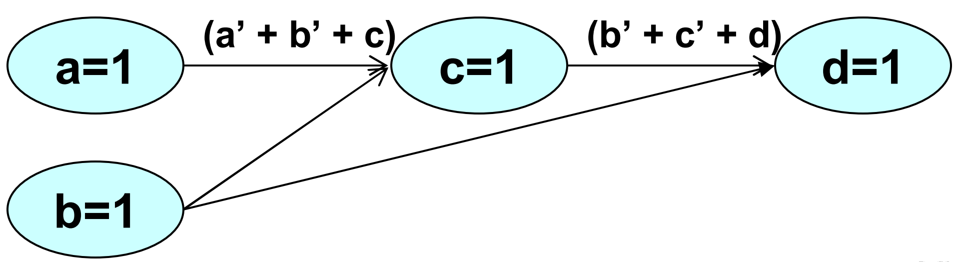

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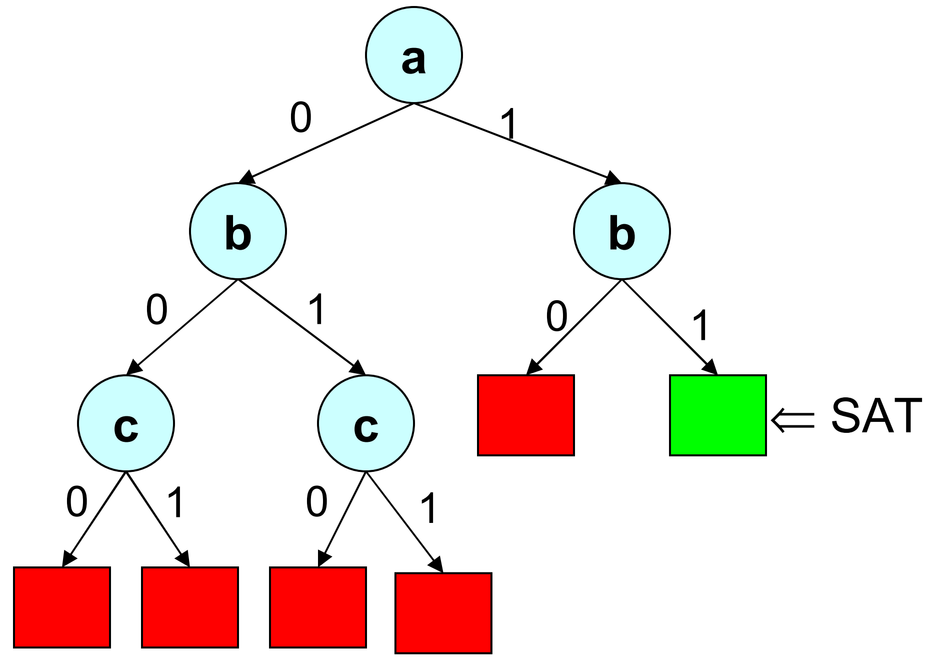
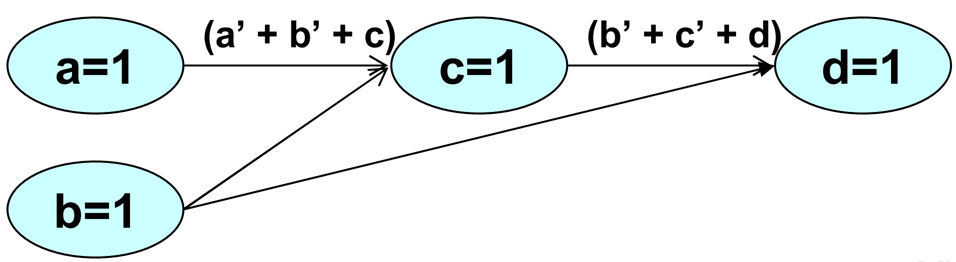




Implications and Boolean Constraint Propagation

- Implication
  - A variable is forced to be assigned to be True or False based on previous assignments.
- Unit clause rule (rule for elimination of one literal clauses)
  - An unsatisfied clause is a unit clause if it has exactly one unassigned literal.

$$(a + b' + c)(b + c')(a' + c')$$

a = T, b = T, c is unassigned

Satisfied Literal

Unsatisfied Literal

Unassigned Literal

  - The unassigned literal is implied because of the unit clause.
- Boolean Constraint Propagation (BCP)
  - Iteratively apply the unit clause rule until there is no unit clause available.
- Workhorse of DLL based algorithms.

Features of DLL

- Eliminates the exponential memory requirements of DP
- Exponential time is still a problem
- Limited practical applicability – largest use seen in automatic theorem proving
- The original DLL algorithm has seen a lot of success for solving random generated instances.

Some Notes

- There are another rules proposed by the original DLL paper, which is seldom used in practice
  - Pure literal rule: if a variable only occur in one phase in the clause database, then the literal can be simply assigned with the value *true*
- The original DP paper also included the unit implication rule to simplify the clauses generated from resolution
  - Still may result in memory explosion
- DLL and DP algorithms are tightly related
  - Fundamentally, both are based on the resolution operation

SAT Algorithm: An Overview

- Davis, Putnam, 1960
  - Explicit resolution based
  - May explode in memory
- Davis, Logemann, Loveland, 1962
  - Search based.
  - Most successful, basis for almost all modern SAT solvers
  - Learning and non-chronological backtracking, 1996
- Stålmarcks algorithm, 1980s
  - Proprietary algorithm. Patented.
  - Commercial versions available
- Stochastic Methods, 1992
  - Unable to prove unsatisfiability, but may find solutions for a satisfying problem quickly.
  - Local search and hill climbing

Stålmarck's Algorithm

M. Sheeran and G. Stålmarck "A tutorial on Stålmarck's proof procedure", *Proc. FMCAD*, 1998

- Algorithm:
  - Using triplets to represent formula
    - Closer to a circuit representation
  - Branch on variable relationships besides on variables
    - Ability to add new variables on the fly
  - Breadth first search over all possible trees in increasing depth

Stålmarck's algorithm (A Vastly Simplified Version)

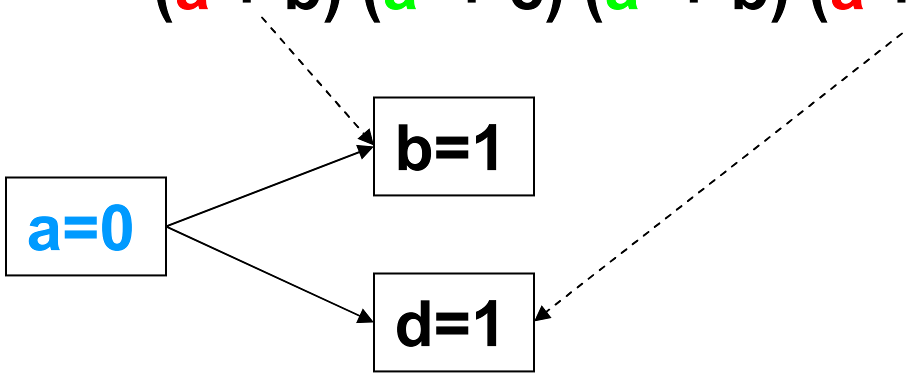
- Try both sides of a branch to find forced decisions (relationships between variables)

$$(a + b)\,(a' + c)\,(a' + b)\,(a + d)$$

Stålmarck’s algorithm (A Vastly Simplified Version)

- Try both sides of a branch to find forced decisions

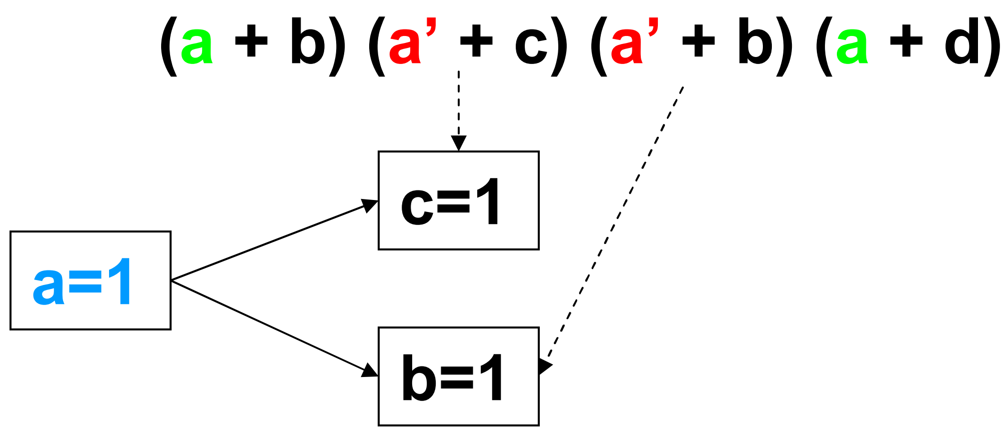
$$(a + b)(a' + c)(a' + b)(a + d)$$



$$\boxed{a=0 \Rightarrow b=1, d=1}$$

Stålmarck's algorithm (A Vastly Simplified Version)

- Try both side of a branch to find forced decisions



$\boxed{a=0 \Rightarrow b=1, d=1}$

$\boxed{a=1 \Rightarrow b=1, c=1}$

Stålmarck's algorithm (A Vastly Simplified Version)

- Try both sides of a branch to find forced decisions

$$(a + b)\ (a' + c)\ (a' + b)\ (a + d)$$

$$\boxed{a=0 \Rightarrow b=1, d=1}$$

$$\boxed{a=1 \Rightarrow b=1, c=1}$$

$$\Rightarrow\ b=1$$

- Repeat for all variables
- Repeat for all pairs, triples,… till either SAT or UNSAT is proved

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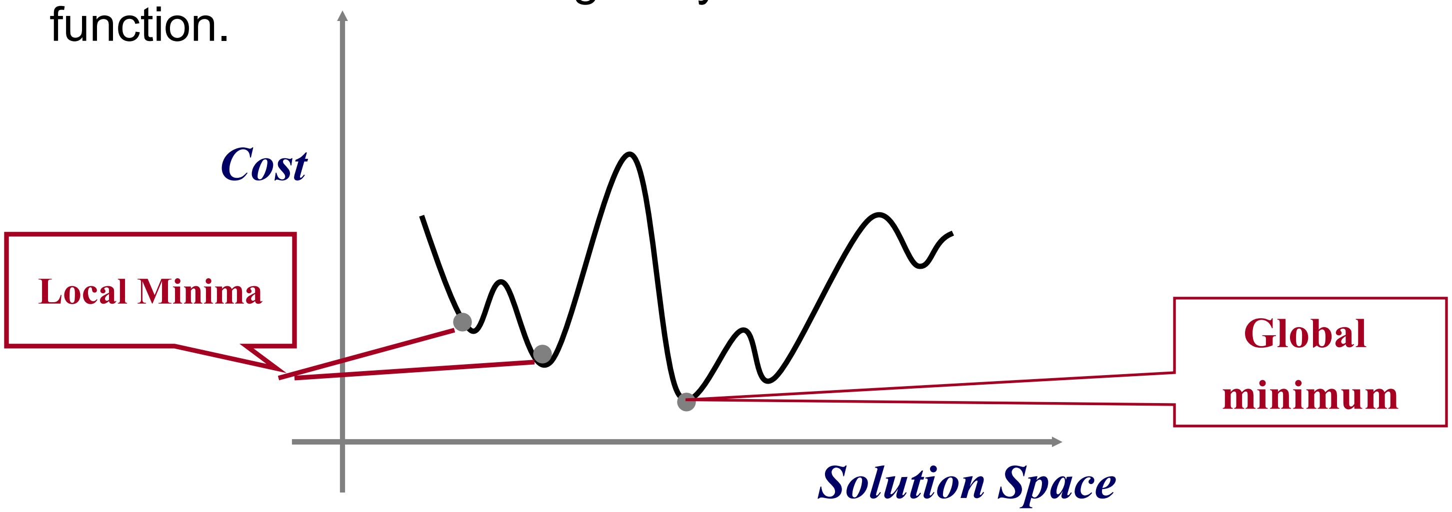
  - Local search and hill climbing

Local Search (GSAT, WSAT)

B. Selman, H. Levesque, and D. Mitchell. "A new method for solving hard satisfiability problems". *Proc. AAAI*, 1992.

- View the solution space as a set of points connected to each other
- There is cost function which needs to be minimized that can be computed for each point.
- Local search involves starting at some point in the solution space, and moving to adjacent points in an attempt to lower the cost function.
- The search is said to be greedy if it does not ever increase the cost function.



Local Search for Max-SAT

- MAX-SAT:
  - Find an assignment that satisfies the most number of clauses
  - Cost function for a given assignment: number of unsatisfied clauses
- Local search has been shown to work well for MAX-SAT
- Cost function for SAT?
  - Can continue to use number of unsatisfied clauses
  - However, only points with a cost function of 0 are of interest

Algorithm of GSAT

```
Procedure GSAT
for i:= 1 to MAX-TRIES
    T:= a randomly genrated truth assignment
    for j:= 1 to MAX-FLIPS
        if T satisfies α then return T
        flip the variable that results in the greatest decrease in the
        number of unsatisfied clauses (decrease ≥ 0)
    end for
end for
return "No satisfying assignment found"
```

- decrease = 0 is referred to as a "sideways" move
- sequence of sideways moves is a "plateau"
- success depends on ability to move between successively lower plateaus

Properties of GSAT

- Seems to work well on randomly generated 3-CNF problems
- Can get stuck in a local minima
- Not guaranteed to be complete

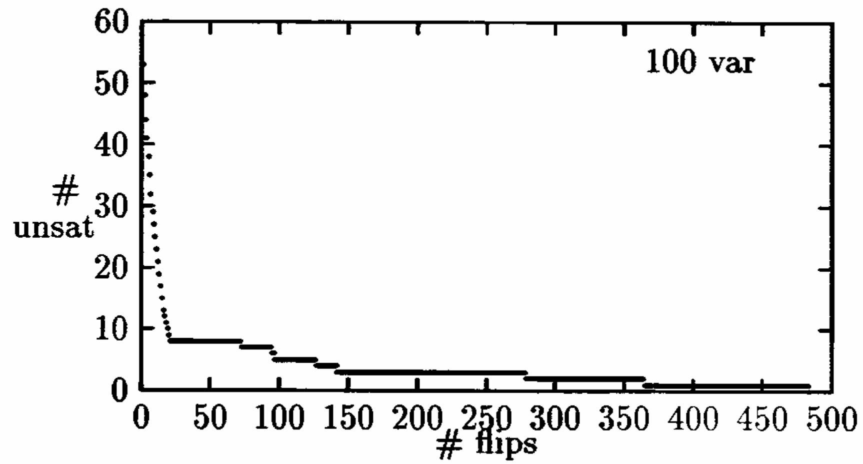


FIGURE 1. GSAT's search space on a randomly-generated 100 variable 3CNF formula with 430 clauses.

Getting out of Local Minima

- Random Walk Strategy

  with probability p, pick a variable occuring in some unsatisfied clause and flip its assignment;

  with probability (1-p), follow the standard GSAT scheme, i.e make the best possible local move
- Random Noise Strategy
  - similar to random walk, except that do not restrict the variable to be flipped to be in an unsatisfied clause
- Simulated Annealing
  - make random flips
  - probabilistically accept “bad moves”

Conclusions about Local Search

- Many local search algorithms exists
  - GSAT, WalkSAT, DLM etc.
  - Differs on how to get out of local minimum
- Incomplete, unable to prove unsatisfiability
  - How to make local search complete is still an open question
- Can be vastly superior than systematic search based algorithms on certain satisfiable formulas
- Has some application in AI planning, limited use in EDA or formal verification