1. 

\[ P(n_h, n_t|\theta) = \theta^{n_h}(1-\theta)^{n_t} \]  

(Binomial likelihood)

\[ \hat{\theta}_{ML} = \frac{n_h}{n_h+n_t} \]  

(Maximum likelihood solution, see last week’s tutorial)

\[ p(\theta) \propto \theta^{\alpha_h-1}(1-\theta)^{\alpha_t-1} \]  

(Beta prior)

\[ E[\theta] = \frac{\alpha_h}{\alpha_h+\alpha_t} \]  

(Prior mean, property of beta distribution)

\[ p(\theta|n_h, n_t) \propto P(n_h, n_t|\theta)p(\theta) \propto \theta^{\alpha_h+n_h-1}(1-\theta)^{\alpha_t+n_t-1} \]  

(Posterior is also a beta distribution)

\[ E[\theta|D] = \frac{\alpha_h+n_h}{\alpha_h+\alpha_t+n_h+n_t} \]  

(Posterior mean, property of beta distribution)

\[ = \frac{\alpha + n}{\alpha + n + \frac{n_h}{n}} \]  

\[ = \frac{\alpha + \frac{n_h}{n}}{\alpha + n \frac{n_h}{n}} \]  

\[ = \lambda E[\theta] + (1-\lambda)\hat{\theta}_{ML} \]  

\( \lambda = \alpha/(\alpha + n) \)

2. 

\[ \frac{d \log p}{d \theta} = \frac{\alpha - 1}{\theta} - \frac{\beta - 1}{1-\theta} \]

\[ \theta^* = \frac{1-\theta^*}{\beta - 1} \] at maximum

\[ \theta^* = \frac{\alpha - 1}{\alpha + \beta - 2} \] is the value of \( \theta \) at maximum

For same positions of mean and mode

\[ \frac{\alpha - 1}{\alpha + \beta - 2} = \frac{\alpha}{\alpha + \beta} \]

\[ \Rightarrow \alpha = \beta \]
3. The graphical model and conditional probabilities are given by.

\[
\begin{align*}
\mathbb{P}[X_n = x] &= \begin{cases} 
\alpha : x = 1 \\
1 - \alpha : x = 2 
\end{cases} \\
\mathbb{P}[S = s] &= \begin{cases} 
\beta : x = 1 \\
1 - \beta : x = 2 
\end{cases} \\
\mathbb{P}[C_n = c | X_n = x, S = s] &= \begin{cases} 
1 - \lambda : c = 1, x = 1, s = 1 \\
\lambda : c = 2, x = 1, s = 1 \\
\delta : c = 1, x = 2, s = 1 \\
1 - \delta : c = 2, x = 2, s = 1 \\
1 - \lambda : c = 2, x = 1, s = 2 \\
\lambda : c = 1, x = 1, s = 2 \\
\delta : c = 1, x = 2, s = 2 \\
\delta : c = 2, x = 2, s = 2 
\end{cases}
\end{align*}
\]

Here \( \alpha = 0.3, \beta = 0.25, \gamma = 0.1, \delta = 0.01. \)

Given data \( c_{1:3} = \{1, 1, 2\} \) probability of next crowd-sourced image label

\[
\begin{align*}
\mathbb{P}[C_4 = 1 | C_{1:3} = c_{1:3}] &= \sum_{x_4 \in \{1, 2\}} \sum_{s \in \{1, 2\}} \mathbb{P}[C_4 = 1, X_4 = x_4, S = s | C_{1:3} = c_{1:3}] \\
&= \sum_{x_4 \in \{1, 2\}} \sum_{s \in \{1, 2\}} \mathbb{P}[C_4 = 1 | X_4 = x_4, S = s, C_{1:3} = c_{1:3}] \mathbb{P}[X_4 = x_4 | S = s, C_{1:3} = c_{1:3}] \mathbb{P}[S = s | C_{1:3} = c_{1:3}] \\
&= \sum_{s \in \{1, 2\}} \left( \sum_{x_4 \in \{1, 2\}} \mathbb{P}[C_4 = 1 | X_4 = x_4, S = s] \mathbb{P}[X_4 = x_4] \mathbb{P}[S = s | C_{1:3} = c_{1:3}] \right)
\end{align*}
\]

Probability of supervisor identity given data

\[
\begin{align*}
\mathbb{P}[S = s | C_{1:3} = \{1, 1, 2\}] &\propto \mathbb{P}[C_{1:3} = \{1, 1, 2\} | S = s] \mathbb{P}[S = s] \\
&= \prod_{n \in \{1, 2, 3\}} \left\{ \mathbb{P}[C_n = c_n | S = s] \right\} \mathbb{P}[S = s] \\
&= \prod_{n \in \{1, 2, 3\}} \left\{ \sum_{x_n \in \{1, 2\}} (\mathbb{P}[C_n = c_n | X_n = x_n, S = s] \mathbb{P}[X_n = x_n]) \right\} \mathbb{P}[S = s]
\end{align*}
\]

2
Substituting in values gives

\[
P[S = 1 \mid C_{1:3} = \{1, 1, 2\}] \propto ((1 - \lambda)\alpha + \delta(1 - \alpha))^2 (\lambda\alpha + \delta(1 - \alpha)) \beta
\]

\[
= 0.277^2 \times 0.723 \times 0.75 = 0.0416
\]

\[
P[S = 2 \mid C_{1:3} = \{1, 1, 2\}] \propto (\lambda\alpha + \delta(1 - \alpha))^2 ((1 - \lambda)\alpha + \delta(1 - \alpha))(1 - \beta)
\]

\[
= 0.723^2 \times 0.277 \times 0.25 = 0.0362
\]

\[
\therefore P[S = 1 \mid C_{1:3} = \{1, 1, 2\}] = \frac{0.0416}{0.0416 + 0.0362} = 0.535 = \epsilon
\]

and \[
P[S = 2 \mid C_{1:3} = \{1, 1, 2\}] = \frac{0.0362}{0.0416 + 0.0362} = 0.465 = 1 - \epsilon
\]

Substituting this into above gives

\[
P[C_4 = 1 \mid C_{1:3} = c_{1:3}] = [(1 - \gamma)\alpha + \delta + \delta(1 - \alpha)]\epsilon + [\gamma\alpha + (1 - \delta)(1 - \alpha)](1 - \epsilon) = 0.485 = \zeta
\]

For final part, the probability of the true label being a car given the now observed crowd-sourced label \(C_4 = 1\)

\[
P[X_4 = 1 \mid C_4 = 1, C_{1:3} = c_{1:3}] = \frac{P[X_4 = 1, C_4 = 1 \mid C_{1:3} = c_{1:3}]}{P[C_4 = 1 \mid C_{1:3} = c_{1:3}]}
\]

\[
= \sum_{s \in \{1, 2\}} \frac{P[X_4 = 1, C_4 = 1, S = s \mid C_{1:3} = c_{1:3}]}{P[C_4 = 1 \mid C_{1:3} = c_{1:3}]}
\]

\[
= \sum_{s \in \{1, 2\}} \frac{P[C_4 = 1 \mid X_4 = 1, S = s, C_{1:3} = c_{1:3}] P[X_4 = 1, S = s \mid C_{1:3} = c_{1:3}]}{P[C_4 = 1 \mid C_{1:3} = c_{1:3}]}
\]

\[
= \frac{P[X_4 = 1 \mid C_4 = 1, C_{1:3} = c_{1:3}]}{P[C_4 = 1 \mid C_{1:3} = c_{1:3}]}
\]

\[
= \frac{\alpha(1 - \gamma)\epsilon + \gamma(1 - \epsilon)}{\zeta} = 0.327
\]