Answers to PMR tutorial questions (Week 3)

Kian Ming Adam Chai and Chris Williams

October 13, 2008

1. Assume that for each box, there is equal probability of choosing any of the fruits in a box (i.e. the probability of choosing a fruit is affected by neither the size, texture, nor smell of the fruit). Also assume equal probability of choosing any box. These two assumptions are/reflects our *prior* knowledge. We have

$$P(\texttt{BOX}=1) = P(\texttt{BOX}=2) = \frac{1}{2}$$

$$P(\texttt{FRUIT}=\texttt{apple}|\texttt{BOX}=1) = \frac{8}{8+4} = \frac{2}{3} \qquad P(\texttt{FRUIT}=\texttt{apple}|\texttt{BOX}=2) = \frac{10}{10+2} = \frac{5}{6}$$

$$P(\texttt{FRUIT}=\texttt{apple}) = \sum_{\texttt{box}\in\{1,2\}} P(\texttt{FRUIT}=\texttt{apple}|\texttt{BOX}=\texttt{box}) \times P(\texttt{BOX}=\texttt{box}) = \frac{2}{3}\times\frac{1}{2} + \frac{5}{6}\times\frac{1}{2} = \frac{3}{4} \qquad \blacksquare$$

$$P(\texttt{BOX}=1|\texttt{FRUIT}=\texttt{apple}) = \frac{P(\texttt{BOX}=1,\texttt{FRUIT}=\texttt{apple})}{P(\texttt{FRUIT}=\texttt{apple})} = \frac{P(\texttt{FRUIT}=\texttt{apple}|\texttt{BOX}=1)\times P(\texttt{BOX}=1)}{P(\texttt{FRUIT}=\texttt{apple})} = \frac{\frac{2}{3}\times\frac{1}{2}}{\frac{3}{4}} = \frac{4}{9} \qquad \blacksquare$$

2. Using the product rule,

$$P(X,Y,Z) = P(Z)P(X,Y|Z) = P(Z)P(X|Z)P(Y|X,Z)$$

Dividing throughout by $P(Z)$ gives $P(X,Y|Z) = P(X|Z)P(Y|X,Z)$.

$$P(X|Y,Z) \stackrel{\text{def}}{=} \frac{P(X,Y,Z)}{P(Y,Z)} = \frac{\cancel{P(Z)}P(X|Z)P(Y|X,Z)}{\cancel{P(Z)}P(Y|Z)} = \frac{P(X|Z)P(Y|X,Z)}{P(Y|Z)}$$

This can be seen as simply $P(X|Y) = \frac{P(X)P(Y|X)}{P(Y)}$ conditional on Z throughout.

3. Given $P(\texttt{DNA match}|\texttt{innocent}) = 10^{-5}$ (False positive rate, FPR), $P(\texttt{DNA match}|\texttt{guilty}) = 1$; assume equal prior probability of anyone of the N in the city committing the crime and that only one person can be guilty, i.e. $P(\texttt{guilty}) = 1/N$, we have

$$\begin{aligned} P(\texttt{DNA match}) &= P(\texttt{DNA match}|\texttt{guilty})\times P(\texttt{guilty}) + P(\texttt{DNA match}|\texttt{innocent})\times P(\texttt{innocent}) \\ &= 1\times N^{-1} + 10^{-5}\times\left(1-N^{-1}\right) = 10^{-5} + (1-10^{-5})N^{-1} \end{aligned}$$

$$P(\texttt{guilty}|\texttt{DNA match}) = \frac{P(\texttt{DNA match}|\texttt{guilty})\times P(\texttt{guilty})}{P(\texttt{DNA match})} = \frac{1\times N^{-1}}{10^{-5}+(1-10^{-5})N^{-1}} = \frac{10^5}{N+99,999}$$

For $N = 10,000$, the required probability is 0.909.

As N increases, $P(\texttt{guilty}|\texttt{DNA match})$ decreases, as the low FPR is multiplied by the size of the population.

4. (Three Prisoners' problem) $\frac{2}{3}$, the event of handing letter to B does not affect A.

Let G be the random variable, taking values on $\{A,B,C\}$, indicating the guilty prisoner. Prior probabilities $P(G=A) = P(G=B) = P(G=C) = \frac{1}{3}$.

Let $I_B \stackrel{\text{def}}{=}$ 'Guard passes letter to B' $\equiv$ 'Guard said that B will be declared innocent'. Note $I_B \in \{\text{true}, \text{false}\}$, i.e. a predicate. We have the following conditionals

$$P(I_B=\text{true}|G=A) = \frac{1}{2} \qquad P(I_B=\text{true}|G=B) = 0 \qquad P(I_B=\text{true}|G=C) = 1$$

so that

$$P(I_B=\text{true}) = \sum_{g\in\{A,B,C\}} P(I_B=\text{true}|G=g)P(G=g) = \left(\frac{1}{2}+0+1\right)\times\frac{1}{3} = \frac{1}{2}$$

Now, conditioned on the fact that $I_B = \texttt{true}$, we have

$$P(G = A | I_B = \texttt{true}) = \frac{P(I_B = \texttt{true} | G = A) P(G = A)}{P(I_B = \texttt{true})} = \frac{\frac{1}{2} \cdot \frac{1}{3}}{\frac{1}{2}} = \frac{1}{3}$$

Hence, probability of release of A = $\frac{2}{3}$ $\blacksquare$.

Also, $P(G = B | I_B = \texttt{true}) = 0$ and $P(G = C | I_B = \texttt{true}) = \frac{2}{3}$.

5. (Monty Hall) Yes. Prize ≡ guilt; open box ≡ giving letter. Prior probabilities

$$P(\texttt{Prize Box} = 1) = P(\texttt{Prize Box} = 2) = P(\texttt{Prize Box} = 3) = \frac{1}{3}$$

Denote the three boxes as `picked box`, `opened box` and `leftover box`.

$$P(\texttt{Opened box is empty} = \texttt{true} | \texttt{Prize Box} = \texttt{picked box}) = \frac{1}{2}$$
$$P(\texttt{Opened box is empty} = \texttt{true} | \texttt{Prize Box} = \texttt{opened box}) = 0$$
$$P(\texttt{Opened box is empty} = \texttt{true} | \texttt{Prize Box} = \texttt{leftover box}) = 1$$

so that

$$P(\texttt{Opened box is empty} = \texttt{true}) = \frac{1}{2}$$

Hence

$$P(\texttt{Prize Box} = \texttt{picked box} | \texttt{Opened box is empty} = \texttt{true}) = \frac{\frac{1}{2} \cdot \frac{1}{3}}{\frac{1}{2}} = \frac{1}{3}$$
$$P(\texttt{Prize Box} = \texttt{leftover box} | \texttt{Opened box is empty} = \texttt{true}) = \frac{1 \cdot \frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$$

Switch, since

$$P(\texttt{Prize Box} = \texttt{leftover box} | \texttt{Opened box is empty} = \texttt{true}) > P(\texttt{Prize Box} = \texttt{picked box} | \texttt{Opened box is empty} = \texttt{true})$$

You can find further discussion of this problem at `http://en.wikipedia.org/wiki/Monty_Hall_problem`.