Hidden Markov Models

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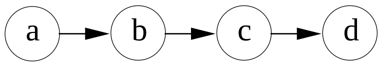
Overview

- Definitions
- Inference Problems
- Recursion formulae
- Viterbi alignment
- Training a HMM
- Reading: Bishop §13.1, 13.2 (but not 13.2.3, 13.2.4, 13.2.5), Rabiner paper

Dynamical models used in many areas for modelling sequences, including

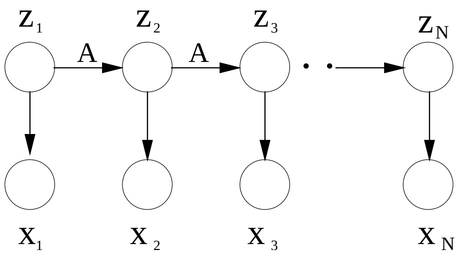
- Speech recognition
- Molecular biology sequences
- Linguistic sequences (e.g. part-of-speech tagging)
- Multi-electrode spike-train analysis
- Tracking objects through time

Markov Chain



$$p(a, b, c, d) = p(a)p(b|a)p(c|b)p(d|c)$$

Hidden Markov Model



A HMM is defined by

- K the number of states
- A the state transition matrix
- $p(\mathbf{x}_n|\mathbf{z}_n)$ the output probability distribution (independent of n)
- π the initial transition probabilities
- $\mathbf{z}_n$ is a multinomial variable with components z_{ni}, if $\mathbf{z}_n$ is in state i then $z_{ni} = 1$ and $z_{nj} = 0$ for $j \neq i$

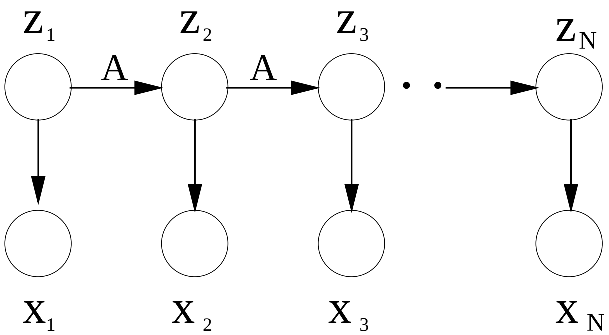
- Let $\pi_{\mathbf{z}_1} = \prod_{i=1}^{K} [\pi_i]^{z_{1i}}$ and

$$a_{\mathbf{z}_n\mathbf{z}_{n+1}} = \prod_{i,j} [a_{ij}]^{z_{ni}\ z_{n+1,j}}$$

- Let $\mathbf{X} = \mathbf{x}_1, \ldots, \mathbf{x}_N$ and $\mathbf{Z} = \mathbf{z}_1, \ldots, \mathbf{z}_N$
- For any state sequence $\mathbf{Z}$

$$p(\mathbf{X}, \mathbf{Z}) = \pi_{\mathbf{z}_1} p(\mathbf{x}_1|\mathbf{z}_1) a_{\mathbf{z}_1\mathbf{z}_2} p(\mathbf{x}_2|\mathbf{z}_2) \ldots a_{\mathbf{z}_{N-1}\mathbf{z}_N} p(\mathbf{x}_N|\mathbf{z}_N)$$

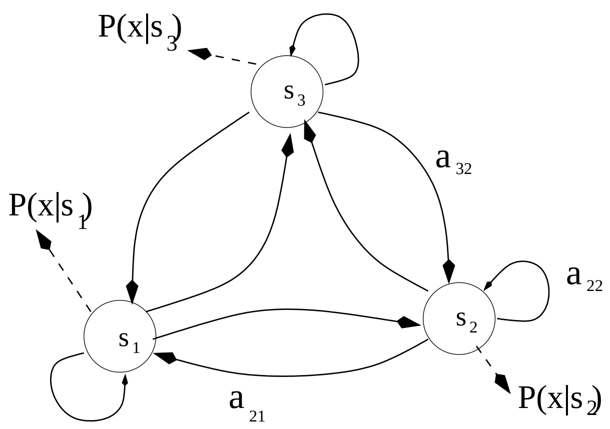
Independence relationships



- Conditioning on $\mathbf{z}_n$ renders $\mathbf{z}_{n-1}$ and $\mathbf{z}_{n+1}$ independent, i.e. $I(\mathbf{z}_{n-1}, \mathbf{z}_{n+1}|\mathbf{z}_n)$
- $I(\mathbf{z}_s, \mathbf{z}_u|\mathbf{z}_n)$ for all $s < n$, $u > n$

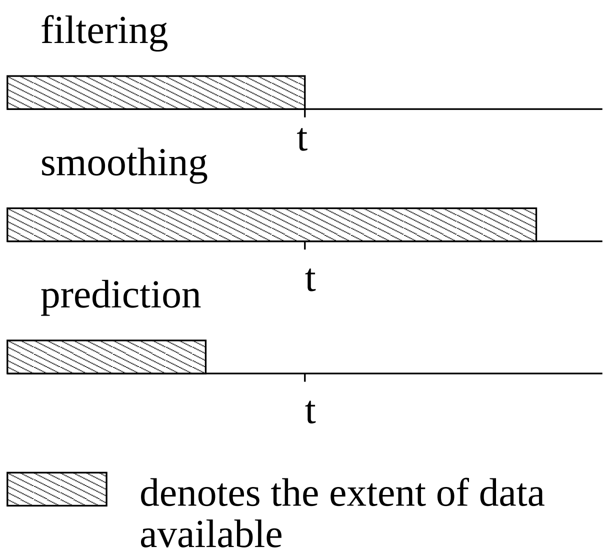
- $I(\mathbf{x}_s, \mathbf{x}_u|\mathbf{z}_n)$ for all $s \leq n$, $u > n$
- the future is independent of the past given the present
- Note that conditioning on $\mathbf{x}_n$ does not yield any conditional independences
- HMM as a dynamical mixture model; choice of state is not independent at each time frame, but depends on the past

- HMM as a finite state automaton

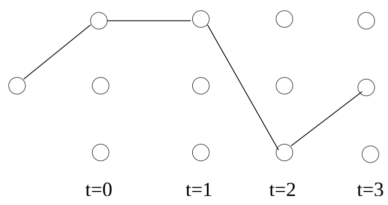


Inference Problems

- $p(\mathbf{z}_1 \dots \mathbf{z}_N|\mathbf{X})$ inferring hidden state given $\mathbf{X}$
- $p(\mathbf{z}_n|\mathbf{X})$ marginal of above
- $p(\mathbf{z}_n|\mathbf{x}_1, \dots, \mathbf{x}_n)$ filtering
- $p(\mathbf{z}_n|\mathbf{x}_1, \dots, \mathbf{x}_s)$ $n > s$, prediction
- $p(\mathbf{z}_n|\mathbf{x}_1, \dots, \mathbf{x}_u)$ $n < u$, smoothing
- $p(\mathbf{x}_1, \dots, \mathbf{x}_N)$ likelihood calculation
- Find sequence $\mathbf{z}_1^* \dots \mathbf{z}_N^*$ that maximizes $p(\mathbf{Z}|\mathbf{X})$ [Viterbi alignment]



$$p(\mathbf{x}_1, \dots, \mathbf{x}_N) = \sum_{\mathbf{z}_1} \sum_{\mathbf{z}_2} \dots \sum_{\mathbf{z}_N} \pi_{\mathbf{z}_1} \prod_{n=1}^{N-1} a_{\mathbf{z}_n \mathbf{z}_{n+1}} \prod_{n=1}^{N} p(\mathbf{x}_n|\mathbf{z}_n)$$



- Naive approach $O(K^N)$
- Efficient $O(K^2 N)$ algorithm is available by pushing sums through products

Computing $p(\mathbf{z}_n|\mathbf{X})$

$$
\begin{aligned}
p(\mathbf{z}_n|\mathbf{X}) &= \frac{p(\mathbf{X}|\mathbf{z}_n)p(\mathbf{z}_n)}{p(\mathbf{X})} \\
&= \frac{p(\mathbf{x}_1,\ldots,\mathbf{x}_n|\mathbf{z}_n)p(\mathbf{x}_{n+1}\ldots\mathbf{x}_N|\mathbf{z}_n)p(\mathbf{z}_n)}{p(\mathbf{X})} \\
&= \frac{p(\mathbf{x}_1,\ldots,\mathbf{x}_n,\mathbf{z}_n)p(\mathbf{x}_{n+1}\ldots\mathbf{x}_N|\mathbf{z}_n)}{p(\mathbf{X})} \\
&\equiv \frac{\alpha(\mathbf{z}_n)\beta(\mathbf{z}_n)}{p(\mathbf{X})}
\end{aligned}
$$

The alphas and betas can be calculated recursively.

$$\sum_{\mathbf{z}_n} p(\mathbf{z}_n|\mathbf{X}) = 1 = \frac{\sum_{\mathbf{z}_n}\alpha(\mathbf{z}_n)\beta(\mathbf{z}_n)}{p(\mathbf{X})}$$

implies

$$p(\mathbf{X}) = \sum_{\mathbf{z}_n}\alpha(\mathbf{z}_n)\beta(\mathbf{z}_n)$$

- Define $p(\mathbf{z}_n|\mathbf{X}) = \gamma(\mathbf{z}_n)$

Recursion formulae

- Alpha

$$\alpha(\mathbf{z}_{n+1}) = \sum_{\mathbf{z}_n}\alpha(\mathbf{z}_n)a_{\mathbf{z}_n\mathbf{z}_{n+1}}p(\mathbf{x}_{n+1}|\mathbf{z}_{n+1})$$

  Initialization

$$\alpha(\mathbf{z}_1) = p(\mathbf{x}_1,\mathbf{z}_1) = p(\mathbf{x}_1|\mathbf{z}_1)p(\mathbf{z}_1) = p(\mathbf{x}_1|\mathbf{z}_1)\pi_{\mathbf{z}_1}$$

- Beta

$$\beta(\mathbf{z}_n) = \sum_{\mathbf{z}_{n+1}}\beta(\mathbf{z}_{n+1})a_{\mathbf{z}_n\mathbf{z}_{n+1}}p(\mathbf{x}_{n+1}|\mathbf{z}_{n+1})$$

  Initialization: $\beta(\mathbf{z}_N)$ is the vector of ones as

$$\sum_i \alpha(z_{Ni})\beta(z_{Ni}) = \sum_i \alpha(z_{Ni}) = \sum_i p(\mathbf{x}_1,\ldots,\mathbf{x}_N,\mathbf{z}_N = i) = p(\mathbf{X})$$

Each step is $O(K^2)$

Viterbi alignment

Find the best state sequence $\mathbf{z}_1^*\mathbf{z}_2^*\ldots\mathbf{z}_N^*$ that maximizes $p(\mathbf{Z}|\mathbf{X})$
Define

$$\delta_n(i) = \max_{\mathbf{z}_1,\ldots\mathbf{z}_{n-1}} p(\mathbf{z}_1,\ldots,\mathbf{z}_n = i,\mathbf{x}_1\ldots\mathbf{x}_n)$$

i.e. $\delta_n(i)$ is the best score along a single path up to time n which account for the first n observations and ends in state S_i.

- There is a recursive formula for the δs similar to the α-recursion, except that a max rather than sum operation is used
- For further details see, for example, L. Rabiner, Proc. IEEE 77(2) 1989 pp 257-285

Calculating $p(\mathbf{z}_n, \mathbf{z}_{n+1}|\mathbf{X})$

$$
\begin{aligned}
\xi(\mathbf{z}_n, \mathbf{z}_{n+1}) &\equiv p(\mathbf{z}_n, \mathbf{z}_{n+1}|\mathbf{X}) \\
&= \frac{p(\mathbf{z}_n, \mathbf{z}_{n+1}, \mathbf{X})}{p(\mathbf{X})} \\
&= \frac{p(\mathbf{X}|\mathbf{z}_n, \mathbf{z}_{n+1})p(\mathbf{z}_{n+1}|\mathbf{z}_n)p(\mathbf{z}_n)}{p(\mathbf{X})} \\
&= p(\mathbf{x}_0, \ldots, \mathbf{x}_n|\mathbf{z}_n)p(\mathbf{x}_{n+1}|\mathbf{z}_{n+1})\times \\
&\quad \frac{p(\mathbf{x}_{n+2} \ldots \mathbf{x}_N|\mathbf{z}_{n+1})p(\mathbf{z}_{n+1}|\mathbf{z}_n)p(\mathbf{z}_n)}{p(\mathbf{X})} \\
&= \frac{\alpha(\mathbf{z}_n)p(\mathbf{x}_{n+1}|\mathbf{z}_{n+1})\beta(\mathbf{z}_{n+1})a_{\mathbf{z}_n\mathbf{z}_{n+1}}}{p(\mathbf{X})}
\end{aligned}
$$

Training a HMM

- Use the EM algorithm to estimate π, A and η, the parameters of $p(\mathbf{x}_n|\mathbf{z}_n)$. Let $\theta = (\pi, A, \eta)$
- If we knew the "true" state sequence, parameter estimation would be easy. The trick is to use the probability distribution over state paths to weight these estimates

$$
\begin{aligned}
\hat{\pi}_i &\leftarrow \gamma(z_{1i}) \\
\hat{a}_{ij} &\leftarrow \frac{\sum_{n=1}^{N-1} \xi(z_{ni}\ z_{n+1,j})}{\sum_{n=1}^{N-1} \gamma(z_{ni})}
\end{aligned}
$$

If the output is a multinomial distribution with $p(x_{nj} = 1|z_{ni} = 1) = \eta_{ij}$ and $\sum_k x_{nk} = 1$

$$
\hat{\eta}_{ij} \leftarrow \frac{\sum_{n=1}^{N} \gamma(z_{ni})x_{nj}}{\sum_{t=1}^{N} \gamma(z_{ni})}
$$

For HMMs these are known as the Baum-Welch equations

Example: Harmonizing Chorales in the Style of J S Bach

- Moray Allan and Chris Williams (NIPS 2004) `http://www.tardis.ed.ac.uk/~moray/harmony/`, online demo at `http://www.anc.inf.ed.ac.uk/demos/hmmbach/`
- Visible states are the melody (quarter notes)
- Hidden states are the harmony (which chord)
- Trained using labelled melody/harmony data (no need for EM)
- Task: find Viterbi alignment for harmony given melody (or sample from $p(\text{harmony}|\text{melody})$.)
- Actually uses HMMs for three subtasks: harmonic skeleton, chord skeleton, ornamentation