The Gaussian Distribution

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Overview

- Probability density functions
- Univariate Gaussian
- Multivariate Gaussian
- Mahalanobis distance
- Properties of Gaussian distributions
- Graphical Gaussian models
- Read: Bishop sec 2.3 (to p 93)

Continuous distributions

- Probability density function (pdf) for a continuous random variable X

$$P(a \leq X \leq b) = \int_a^b p(x)dx$$

therefore

$$P(x \leq X \leq x + \delta x) \simeq p(x)\delta x$$

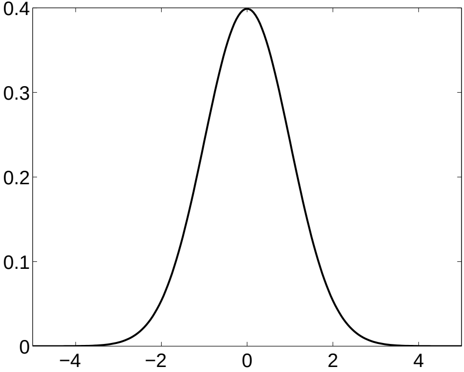
- **Example**: Gaussian distribution

$$p(x) = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp - \left\{ \frac{(x-\mu)^2}{2\sigma^2} \right\}$$

shorthand notation $X \sim N(\mu, \sigma^2)$

- Standard normal (or Gaussian) distribution $Z \sim N(0, 1)$
- Normalization

$$\int_{-\infty}^{\infty} p(x)dx = 1$$



- Cumulative distribution function

$$\Phi(z) = P(Z \leq z) = \int_{-\infty}^{z} p(z')dz'$$

- Expectation

$$E[g(X)] = \int g(x)p(x)dx$$

- mean, $E[X]$
- Variance $E[(X-\mu)^2]$
- For a Gaussian, mean $= \mu$, variance $= \sigma^2$
- Shorthand: $x \sim N(\mu, \sigma^2)$

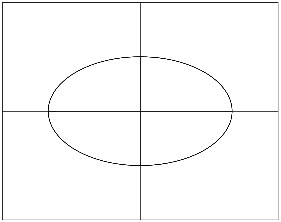
Bivariate Gaussian I

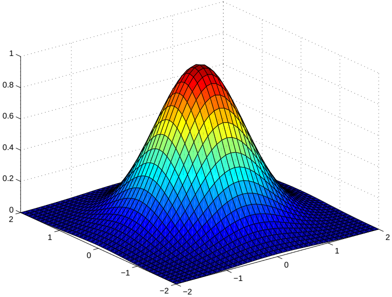
- Let $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$
- If X_1 and X_2 are independent

$$p(x_1, x_2) = \frac{1}{2\pi(\sigma_1^2\sigma_2^2)^{1/2}} \exp -\frac{1}{2}\left\{\frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right\}$$

- Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$, $\Sigma = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}$

$$p(\mathbf{x}) = \frac{1}{2\pi|\Sigma|^{1/2}} \exp -\frac{1}{2}\left\{(\mathbf{x}-\boldsymbol{\mu})^T\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})\right\}$$




1
0.8
0.6
0.4
0.2
0
2
1
0
−1
−2
−2
−1
0
1
2

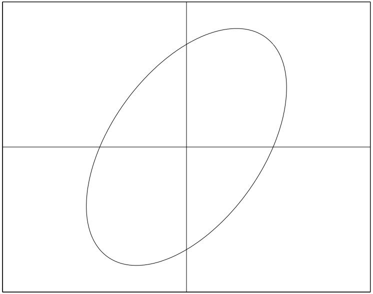
Bivariate Gaussian II

- Covariance
- Σ is the covariance matrix

$$\Sigma = E[(\mathbf{x} - \boldsymbol{\mu})(\mathbf{x} - \boldsymbol{\mu})^T]$$

$$\Sigma_{ij} = E[(x_i - \mu_i)(x_j - \mu_j)]$$

- Example: plot of weight vs height for a population



Multivariate Gaussian

- $P(\mathbf{x} \in \mathcal{R}) = \int_{\mathcal{R}} p(\mathbf{x}) d\mathbf{x}$
- Multivariate Gaussian

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{d/2}|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T \Sigma^{-1} (\mathbf{x}-\boldsymbol{\mu})\right\}$$

- Σ is the covariance matrix

$$\Sigma = E[(\mathbf{x}-\boldsymbol{\mu})(\mathbf{x}-\boldsymbol{\mu})^T]$$

$$\Sigma_{ij} = E[(x_i - \mu_i)(x_j - \mu_j)]$$

- Σ is symmetric
- Shorthand $\mathbf{x} \sim N(\boldsymbol{\mu}, \Sigma)$
- For $p(\mathbf{x})$ to be a density, Σ must be positive definite
- Σ has $d(d+1)/2$ parameters, the mean has a further d

Mahalanobis Distance

$$d_\Sigma^2(\mathbf{x}_i, \mathbf{x}_j) = (\mathbf{x}_i - \mathbf{x}_j)^T \Sigma^{-1} (\mathbf{x}_i - \mathbf{x}_j)$$

- $d_\Sigma^2(\mathbf{x}_i, \mathbf{x}_j)$ is called the Mahalanobis distance between $\mathbf{x}_i$ and $\mathbf{x}_j$
- If Σ is diagonal, the contours of d_Σ^2 are axis-aligned ellipsoids
- If Σ is not diagonal, the contours of d_Σ^2 are *rotated* ellipsoids

  $$\Sigma = U \Lambda U^T$$

  where Λ is diagonal and U is a rotation matrix
- Σ is positive definite $\Rightarrow$ entries in Λ are positive

Parameterization of the covariance matrix

- Fully general $\Sigma \implies$ variables are correlated
- Spherical or isotropic. $\Sigma = \sigma^2 I$. Variables are independent
- Diagonal $[\Sigma]_{ij} = \delta_{ij}\sigma_i^2$ Variables are independent
- Rank-constrained: $\Sigma = WW^T + \Psi$, with W being a $d \times q$ matrix with $q < d - 1$ and Ψ diagonal. This is the factor analysis model. If $\Psi = \sigma^2 I$, then with have the probabilistic principal components analysis (PPCA) model

Transformations of Gaussian variables

- Linear transformations of Gaussian RVs are Gaussian

$$\mathbf{X} \sim N(\boldsymbol{\mu}_x, \Sigma)$$
$$\mathbf{Y} = A\mathbf{X} + \mathbf{b}$$
$$\mathbf{Y} \sim N(A\boldsymbol{\mu}_x + \mathbf{b}, A\Sigma A^T)$$

- Sums of Gaussian RVs are Gaussian

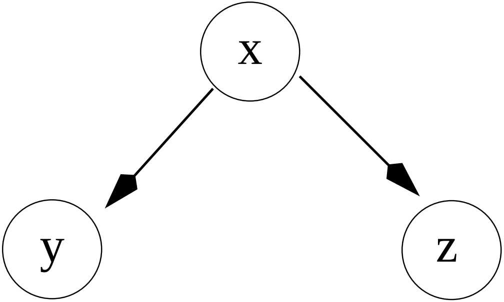
$$Y = X_1 + X_2$$
$$E[Y] = E[X_1] + E[X_2]$$
$$\mathrm{var}[Y] = \mathrm{var}[X_1] + \mathrm{var}[X_2] + 2\mathrm{covar}[X_1, X_2]$$
if X_1 and X_2 are independent $\mathrm{var}[Y] = \mathrm{var}[X_1] + \mathrm{var}[X_2]$

Properties of the Gaussian distribution

- Gaussian has relatively simple analytical properties
- Central limit theorem. Sum (or mean) of M independent random variables is distributed normally as $M \to \infty$ (subject to a few general conditions)
- Diagonalization of covariance matrix $\implies$ rotated variables are independent
- All marginal and conditional densities of a Gaussian are Gaussian
- The Gaussian is the distribution that maximizes the entropy $H = -\int p(\mathbf{x}) \log p(\mathbf{x}) d\mathbf{x}$ for fixed mean and covariance

Graphical Gaussian Models

Example:



- Let X denote pulse rate
- Let Y denote measurement taken by machine 1, and Z denote measurement taken by machine 2

- Model

  $X \sim N(\mu_x, v_x)$
  $Y = \mu_y + w_y(X - \mu_x) + N_y$
  $Z = \mu_z + w_z(X - \mu_x) + N_z$
  noise $N_y \sim N(0, v_y^N)$, $N_z \sim N(0, v_z^N)$, independent

- (X, Y, Z) is jointly Gaussian; can do inference for X given $Y = y$ and $Z = z$

As before

$$P(x, y, z) = P(x)P(y|x)P(z|x)$$

Show that

$$\boldsymbol{\mu} = \begin{pmatrix} \mu_x \\ \mu_y \\ \mu_z \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} v_x & w_y v_x & w_z v_x \\ w_y v_x & w_y^2 v_x + v_y^N & w_y w_z v_x \\ w_z v_x & w_y w_z v_x & w_z^2 v_x + v_z^N \end{pmatrix}$$

Inference in Gaussian models

- Partition variables into two groups, $\mathbf{X}_1$ and $\mathbf{X}_2$

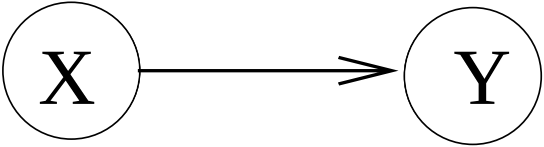
$$\boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$$

$$\boldsymbol{\mu}^c_{1|2} = \boldsymbol{\mu}_1 + \Sigma_{12}\Sigma_{22}^{-1}(\mathbf{x}_2 - \boldsymbol{\mu}_2)$$
$$\Sigma^c_{1|2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$

- For proof see §2.3.1 of Bishop (2006) (not examinable)
- Formation of joint Gaussian is analogous to formation of joint probability table for discrete RVs. Propagation schemes are also possible for Gaussian RVs

Example Inference Problem



$$Y = 2X + 8 + N_y$$

- Assume $X \sim N(0, 1/\alpha)$, so $w_y = 2$, $\mu_y = 8$, and $N_y \sim N(0, 1)$
- Show that

$$\mu_{x|y} = \frac{2}{4+\alpha}(y-8)$$
$$\mathrm{var}(x|y) = \frac{1}{4+\alpha}$$

Hybrid (discrete + continuous) networks

- Could discretize continuous variables, but this is ugly, and gives large CPTs
- Better to use parametric families, e.g. Gaussian
- Works easily when continuous nodes are children of discrete nodes; we then obtain a *conditional Gaussian* model