

Performance Modelling — Lecture 9

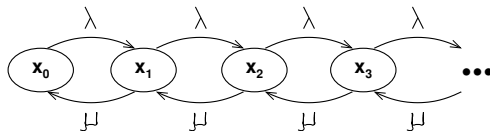
Solving Queueing Models

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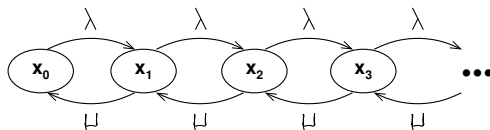
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If we write the global balance equations for this system we can soon recognise a regular pattern emerging.

$$\begin{aligned} \lambda\pi_0 &= \mu\pi_1 \\ (\lambda + \mu)\pi_1 &= \lambda\pi_0 + \mu\pi_2 \\ (\lambda + \mu)\pi_2 &= \lambda\pi_1 + \mu\pi_3 \\ &\vdots \end{aligned}$$

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Using simple algebra we can rewrite these:

$$\begin{aligned}\pi_1 &= \frac{\lambda}{\mu}\pi_0 \\ \pi_2 &= \frac{\lambda}{\mu}\pi_1 = \left(\frac{\lambda}{\mu}\right)^2\pi_0 \\ \pi_3 &= \frac{\lambda}{\mu}\pi_2 = \left(\frac{\lambda}{\mu}\right)^3\pi_0 \\ &\vdots\end{aligned}$$

$M/M/1$: state probabilities

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Recalling that $\lambda/\mu = \rho$, the **traffic intensity**, we can see that for an arbitrary state x_i :

$$\pi_i = \rho^i \pi_0$$

$M/M/1$: normalisation condition

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Substituting the expression for π_i we get:

$$\sum_{i=0}^{\infty} \pi_i = \pi_0 \sum_{i=0}^{\infty} \rho^i = \pi_0 \frac{1}{1-\rho} \quad \text{if } \rho < 1$$

$M/M/1$: steady state probabilities

From the normalisation condition we can deduce:

$$\pi_0 = 1 - \rho$$

and it follows that the probability of being in an arbitrary state i is

$$\pi_i = (1 - \rho)\rho^i \quad \text{for all } i > 0.$$

$M/M/1$: symbolic evaluation

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We do not need to carry out a numerical solution of the global balance equations.

Moreover, from this steady state distribution we can derive required performance measures directly in terms of ρ .

$M/M/1$: Utilisation, U

The queue is being utilised whenever it is non-empty; in other words the utilisation, U , is $1 - \pi_0$.

Utilisation

$$U = \rho$$

$M/M/1$: Mean number of customers in the queue, N

This is the expectation of the number of customers in the service facility as a whole, i.e.

$$N = \sum_{n=1}^{\infty} n \times \pi_n = \sum_{n=1}^{\infty} n \times (1 - \rho)\rho^n = \frac{\rho}{(1 - \rho)}$$

No. in queue

$$N = \frac{\rho}{1 - \rho}$$

$M/M/1$: Mean number of customers waiting, N_b

This is the expectation of the number of customers in the buffer

$$\begin{aligned} N_b &= \sum_{n=1}^{\infty} (n-1) \times \pi_n = \sum_{n=1}^{\infty} (n-1) \times (1-\rho)\rho^n \\ &= \rho \sum_{n=1}^{\infty} n \times (1-\rho)\rho^n = \frac{\rho^2}{(1-\rho)} \end{aligned}$$

Number in buffer

$$N_b = \frac{\rho^2}{1-\rho}$$

$M/M/1$: Mean response time, R

Using Little's Law we can calculate the mean response time of the queue to be the mean number in the queue N , divided by the arrival rate λ ,

$$R = N/\lambda = \frac{\rho}{1-\rho} \times \frac{1}{\lambda} = \frac{1/\mu}{1-\rho} = \frac{1}{\mu(1-\rho)}.$$

Response time

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Thus, given almost any single queue model, in order to derive a performance measure it is only necessary to select the appropriate formula from a table and evaluate it using the parameters of your model.

Some examples are in Lecture Note 9 and most textbooks on performance models will contain these formulae.

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- The gateway takes **2 milliseconds** on average to transmit a packet.
- The gateway currently has **13 places** (including the packet being transmitted) and packets that arrive when the buffer is full are lost.
- We wish to find out if the buffer capacity is sufficient to ensure that **less than one packet per million gets lost**.

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- We represent the gateway as a $M/M/1/13$ queue, with $\lambda = 125$ and $\mu = 1/0.002 = 500$.

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- The loss rate is the arrival rate multiplied by the probability that the system is full, i.e. $\lambda \times \pi_K$.
- The proportion of lost packets is π_K (expected number lost per time unit divided by the expected number arriving per time unit)

$$\begin{aligned}\pi_K &= \frac{(1 - \rho)\rho^K}{1 - \rho^{K+1}} = (0.75 \times 0.25^{13}) / (1 - 0.25^{14}) \\ &= (1.1176 \times 10^{-8}) / 0.99999999 = 1.12 \times 10^{-8}\end{aligned}$$

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- Thus the expected proportion of packets lost is **0.0112 every million packets**, well within the requirement.

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So we would not expect to derive symbolic steady state distributions of wide applicability in the same way as we have done for single queues.

However, for a large class of queueing networks a straightforward and efficient means of solving models has been found. These networks are known as [product form networks](#).

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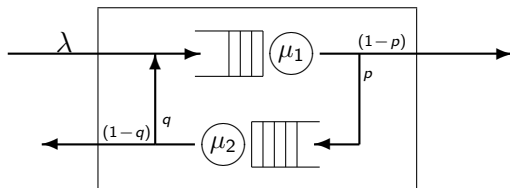
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For example, in a simple queueing network with two service centres, the state **(n_1, n_2)** indicates that there are **n_1 customers in service centre 1** (queueing or in service) and **n_2 customers in service centre 2**.

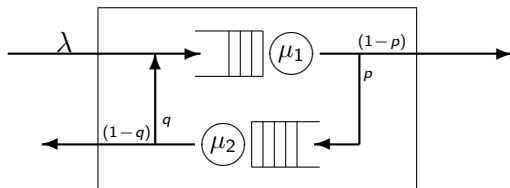
Product form distributions



For this model to have a product form steady state distribution means that the distribution can be expressed as a product of terms representing the steady states of each of the service centres considered in isolation, e.g.

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At steady state the service centres behave **independently**.

Traffic Equations

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The **effective arrival rate** at each queue must be derived taking into consideration any **external arrivals** and **arrivals from other queues**.

A series of **established theorems**, the decomposition principle of Poisson/exponential distributions, and simple algebra, can help us to work out the arrival rate at each node in the network.

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This result implies that each service centre in a chain of simple exponential service centres with Poisson arrivals can be analysed independently using results from simple queues.

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Again, this implies that each service centre in the queueing network can be **analysed independently**.

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Gordon and Newell's product form equation has the form:

$$\pi(n_1, n_2, \dots, n_k) = \frac{1}{G(K)} \prod_{i=1}^n \pi_i(n_i)$$

where $G(K)$ is a **normalisation constant** chosen to ensure that the steady state probabilities sum to 1.

Traffic equations: implications of these results

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By the decomposition principle we know that if the departure stream is split and only goes to queue $i + 1$ with probability p , then the arrival rate at queue $i + 1$ will be $\lambda_{i+1} = p \times \lambda_i$

Traffic equations

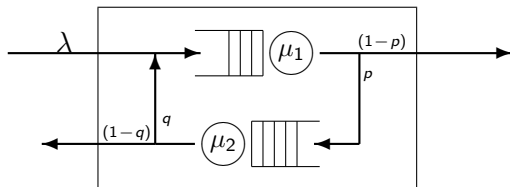
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If there are n service centres, we will have n equations in n unknown and solving the traffic equations we can find the arrival rate at each service centre.

Traffic equations: example

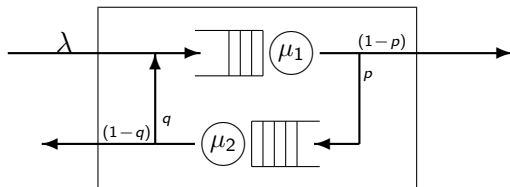


For this simple network the traffic equations are:

$$\lambda_1 = \lambda + q \times \lambda_2$$

$$\lambda_2 = p \times \lambda_1$$

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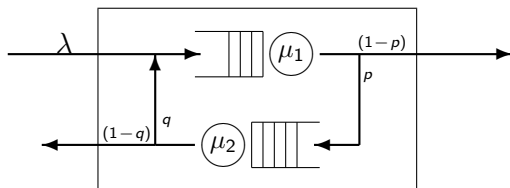
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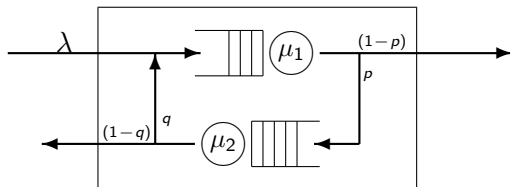
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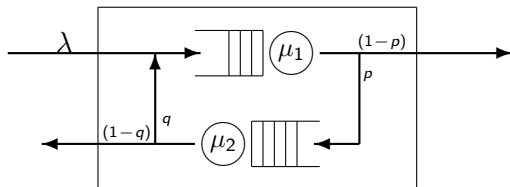
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Therefore $\lambda_1 = 12.5$ and $\lambda_2 = 6.25$.

Traffic equations vs. Global balance equations

The great advantage of solving traffic equations rather than the global balance equations is that the number of equations we need to solve grows **linearly** with the number of service centres, rather than **exponentially**, which is the case for global balance equations.

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One exception is that we can now consider models with an **infinite number of state** since we do not need to numerically solve the global balance equations.

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- The system has **device homogeneity** — the service rate of customers at a service centre may depend on the number of jobs at that centre, but not more generally on the placement of customers within the network.
- The system experiences **homogeneous external arrivals** — the times at which arrivals from outside the network occur may not depend on the number or placement of customers within the network.