Performance Modelling — Lecture 7
Stochastic Petri Nets

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As discussed in a previous lecture, high level modelling formalisms are used to make the job of constructing the state transition diagram and/or infinitesimal generator matrix easier. The second such language we consider is Stochastic Petri Nets.
Petri Nets

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Petri nets were introduced in the 1960s for modelling a variety of concurrent systems, but their use for performance modelling originates from 1980s.
Basic Definitions

The primitives of the notation are the following:

PLACES

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- **PLACES**
  Places are used to represent conditions or local system states, e.g. a place may relate to one phase in the behaviour of a particular component.

- **TRANSITIONS**
  Transitions are used to describe events that occur in the system; these will usually result in a modification to the system state.
Basic Definitions

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Arcs specify the relationships between local states or conditions (places) and events (transitions). An arc from a place to a transition is termed an input arc. This indicates the local state in which the event can occur. An arc to a place from a transition is termed an output arc.
Anatomy of a Petri net

- **Place** ($P_1$, $P_2$, $P_3$)
- **Token**
- **Input arc**
- **Output arc**
- **Transition** ($T_1$)
Example firing

When a transition **fires** tokens from input places are absorbed and tokens are created on each of the output places.
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This is instantaneous.
If a **multiplicity** is associated with an arc, the firing rule is adjusted to reflect the multiplicity by altering the number of tokens **required** or **produced**.

For example, the net above **cannot fire**.
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Petri net marking

The state of the Petri net system at any time, is characterised by the distribution of tokens over the places, generally termed a marking: \( m : P \rightarrow \mathbb{N}_0 \), where \( m(p) = n \) means that there are \( n \) tokens on place \( p \).
The firing rule more formally

A transition $t$ is said to be **enabled** in a marking $m$, written as $m[t]$, if all the **pre-places** of $t$ (those connected by an input arc) have a marking that is greater than or equal to the multiplicity of that input arc.
The firing rule more formally

A transition $t$ is said to be **enabled** in a marking $m$, written as $m[t]$, if all the **pre-places** of $t$ (those connected by an input arc) have a marking that is greater than or equal to the multiplicity of that input arc.

Otherwise $t$ is said to be **disabled**.
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When $t$ in $m$ fires, a new marking $m'$ is reach, written as $m[t]m'$.
Reachability

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If, when playing the token game, as well as all the states we come across, we record the transitions between those states, we obtain the reachability graph.
Example: reachability graph
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\[ P_1 \xrightarrow{T_1} P_2 \xrightarrow{T_1} P_3 \]

Transition labels:
- \((2, 2, 1)\)
- \((3, 2, 0)\)
Example: reachability graph
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\[ P_1 \quad T_1 \quad P_2 \quad T_1 \quad T_2 \quad P_3 \]

\[ (3, 2, 0) \quad T_1 \quad (2, 2, 1) \quad T_2 \quad (2, 2, 0) \]

\[ (2, 2, 0) \quad T_1 \quad (1, 2, 2) \]

\[ (1, 2, 2) \quad T_2 \quad (0, 2, 0) \quad T_2 \quad (0, 2, 1) \quad T_2 \quad (0, 2, 0) \]
Example: reachability graph

- States: $P_1$, $P_2$, $P_3$, $T_1$, $T_2$
- Transitions: $T_1$, $T_2$
- Initial marking: $(3, 2, 0)$
- Transitions:
  - $T_1$: $(3, 2, 0) \rightarrow (2, 2, 1)$
  - $T_1$: $(2, 2, 1) \rightarrow (2, 2, 0)$
  - $T_1$: $(2, 2, 0) \rightarrow (1, 2, 2)$
  - $T_1$: $(1, 2, 2) \rightarrow (0, 2, 3)$
Example: reachability graph

\begin{itemize}
\item \( P_1 \)
\item \( T_1 \)
\item \( P_2 \)
\item \( P_3 \)
\item \( T_2 \)
\end{itemize}

\begin{align*}
(3, 2, 0) & \xrightarrow{T_1} (2, 2, 1) & \xrightarrow{T_2} (2, 2, 0) \\
(2, 2, 1) & \xrightarrow{T_1} (1, 2, 2) & \xrightarrow{T_2} (1, 2, 1) \\
(1, 2, 2) & \xrightarrow{T_1} (0, 2, 3) \\
(0, 2, 3) & \xrightarrow{T_1} (0, 2, 3)
\end{align*}
Example: reachability graph

\[ P_1 \]
\[ T_1 \]
\[ P_2 \]
\[ P_3 \]
\[ T_2 \]

\[ (3, 2, 0) \]
\[ T_1 \]
\[ (2, 2, 1) \]
\[ T_2 \]
\[ (1, 2, 2) \]
\[ T_1 \]
\[ (0, 2, 3) \]
\[ T_2 \]
Example: reachability graph

\[
\begin{align*}
P_1 & \quad T_1 \quad P_2 \\
(3, 2, 0) & \xrightarrow{T_1} (2, 2, 1) \xrightarrow{T_2} (2, 2, 0) \\
(2, 2, 1) & \xrightarrow{T_1} (1, 2, 2) \xrightarrow{T_2} (1, 2, 1) \xrightarrow{T_2} (1, 2, 0) \\
(1, 2, 2) & \xrightarrow{T_1} (0, 2, 3) \xrightarrow{T_2} (0, 2, 2)
\end{align*}
\]
Example: reachability graph

\[
\begin{align*}
P_1 & \rightarrow T_1 & (3, 2, 0) & T_1 \\
T_1 & \rightarrow P_2 & (2, 2, 1) & T_2 \\
P_2 & \rightarrow T_1 & (1, 2, 2) & T_2 \\
P_3 & \rightarrow T_2 & (0, 2, 3) & T_2 \\
T_2 & & & \\
\end{align*}
\]
Example: reachability graph
Stochastic Petri Nets

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In a stochastic Petri nets we associate an **exponentially distributed delay** with the firing of each **transition**.

The delay occurs between when the transition becomes **enabled** and when it **fires**; indeed the **instantaneous** firing will only occur if the transition has remained enabled throughout the delay period.

It is straightforward to show that the **reachability graph** of a SPN forms the **state transition diagram** of an underlying Markov process.
Multi-processor example as a SPN

We consider again the simple multi-processor system, initially just with one processor.

We make a slight modification to the system: we now explicitly represent the processor requesting and gaining access to the common memory.
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A processor executes locally for some time (mean duration $1/\lambda$), and then requests access to common memory (gaining access has mean duration $1/r$). Once it has gained access, the duration of common memory access is assumed to be $1/\mu$ on average.
Multi-processor example as a SPN

- $P_1$: executing in private memory
- $T_1$: requesting
- $P_2$: accessing
- $P_3$: common memory
- $P_4$:
Multi-processor example as a SPN

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- $P_1$ represents the processor when it is executing privately;
- $P_2$ represents the processor when it is ready to access the common memory;
- $P_3$ represents the state when the process is using the common memory;
- $P_4$ represents the common memory when it is not in use;
Multi-processor example as a SPN

- $T_1$ represents the event of the processor executing privately; the rate of this transition is $\lambda$;
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- \( T_1 \) represents the event of the processor executing privately; the rate of this transition is \( \lambda \);
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Multi-processor example as a SPN

- $T_1$ represents the event of the processor executing privately; the rate of this transition is $\lambda$;
- $T_2$ represents the processor gaining access to the common memory; the rate of this transition is $r$;
- $T_3$ represents the processor accessing the common memory; the rate of this transition is $\mu$. 
Multi-processor example as a SPN

The reachability set of this model is \{(1, 0, 0, 1), (0, 1, 0, 1), (0, 0, 1, 0)\},
Multi-processor example as a SPN

The reachability set of this model is \{(1,0,0,1), (0,1,0,1), (0,0,1,0)\}, and the reachability graph is
The two processor system

![Stochastic Petri Net Diagram]

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The two processor system

Now we see why the action of gaining access needs to be explicitly represented in the SPN model. Without it we could not enforce the necessary **mutual exclusion** between the processors.
The two processor system

The reachability set is now

\[ \{ (1,0,0,1,1,0,0), (1,0,0,1,0,1,0), (1,0,0,0,0,0,1), \\
    (0,1,0,1,1,0,0), (0,1,0,1,0,1,0), (0,1,0,0,0,0,1), \\
    (0,0,1,0,1,0,0), (0,0,1,0,0,1,0) \} \]
The two processor system

The reachability set is now

\[ \{(1,0,0,1,1,0,0), (1,0,0,1,0,1,0), (1,0,0,0,0,0,1), (0,1,0,1,1,0,0), (0,1,0,1,0,1,0), (0,1,0,0,0,0,1), (0,0,1,0,1,0,0), (0,0,1,0,0,1,0)\} \]

Note, we have eight distinct states rather than the five states we had in the Markov process when we modelled the system directly.
The two processor system
Generalised Stochastic Petri Nets

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One of the reasons for this is that actions which would not be explicitly represented if we were working directly at the Markov process level have to be represented by transitions.
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One of the reasons for this is that actions which would not be explicitly represented if we were working directly at the Markov process level have to be represented by transitions.

This has the effect of increasing the state space since we now have to consider the state of waiting for these actions to occur, whereas we would prefer to abstract away from these actions.
Generalised Stochastic Petri Nets (GSPN) represent an extension of the SPN formalism, which is designed to address this problem.
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Two new primitives are added to the notation, all others remaining the same and keeping the same interpretation. These new primitives are immediate transitions and inhibitor arcs.
Immediate Transitions

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When they are enabled in a model they fire immediately, and they have priority over any enabled timed transitions.

If two or more immediate transitions can be enabled at the same time, the probability that each of them is the one to fire must be declared in the model.
Immediate Transitions

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Logical actions arise when the system makes a choice between two or more alternatives. In an SPN these actions must be represented as actions which take time. However it is more natural to think of them occurring instantaneously.
Multi-processor example again
Multi-processor example again
Multi-processor example again

\[ \text{Graphical representation of a multi-processor system with transitions and places.} \]

\[ \begin{align*}
(0,0,1,0,1,0,0) & \xrightarrow{T_2} (0,0,1,0,0,1,0) \\
(0,1,0,1,1,0,0) & \xrightarrow{T_4} (1,0,0,1,1,0,0) \\
(0,0,1,0,1,0,0) & \xrightarrow{T_3} (1,0,0,1,0,1,0) \\
(0,1,1,0,1,0,0) & \xrightarrow{T_4} (1,0,0,1,0,1,0) \\
(0,1,0,0,0,0,1) & \xrightarrow{T_6} (1,0,0,0,0,0,1) \\
\end{align*} \]
Multi-processor example again
Multi-processor example again
Multi-processor example again
Multi-processor example again

\[
\begin{align*}
(P_1, T_1, P_2, T_2, P_3, T_3) &= (1, 0, 0, 1, 0, 0) \\
(P_3, T_4, P_6, T_5, P_7, T_6) &= (0, 0, 1, 0, 0, 1) \\
T_3 &\rightarrow (1, 0, 0, 1, 0, 0) \\
T_1 &\rightarrow (0, 0, 1, 0, 0, 1) \\
T_4 &\rightarrow (0, 0, 1, 0, 0, 1) \\
T_6 &\rightarrow (1, 0, 0, 0, 0, 1)
\end{align*}
\]
Multi-processor example again

```
1 1 2 2 3 4
75645
3 6
P T P T P
P
PTPTP
T
T
```

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Inhibitor Arcs

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The inhibitor arcs impose additional constraints to the usual firing rule.
Example with inhibitor arc

Exercise:
Construct the reachability graph for this GSPN with and without the inhibitor arc.
Example with inhibitor arc

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