Performance Modelling — Lecture 4
More Complex Markov Processes

Jane Hillston
School of Informatics
The University of Edinburgh
Scotland

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Upgrading a PC LAN

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The transmission medium supports no more than one transmission at any given time. To resolve conflicts, a token is passed round the network from one node to another in round robin order.

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Other nodes on the network might be peripheral devices such as printers or faxes but for the purposes of this study we make no distinction and assume that all nodes are PCs.

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Our task is to find out how the delay experienced by data packets at each PC will be affected if another two PCs are added.

Modelling Assumptions

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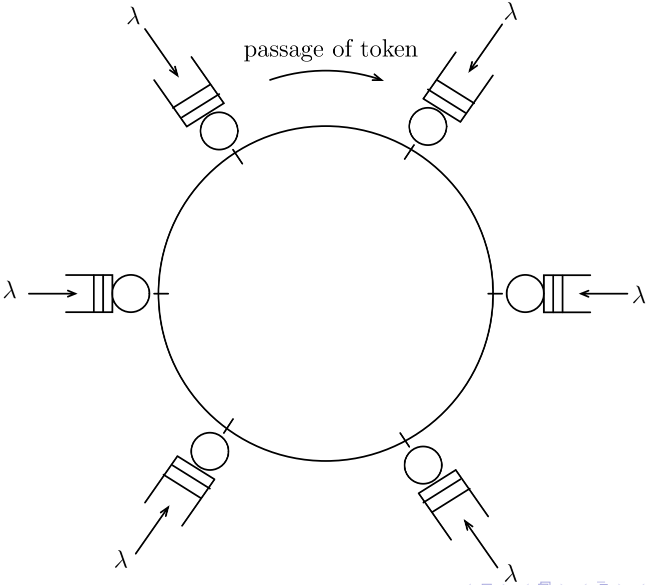
We also know the mean duration, d, of a data packet transmission, and the mean time, m, taken for the token to pass from one PC to the next.

It is assumed that if another data packet is generated, whilst the PC is transmitting, this second data packet must wait for the next visit of the token before it can be transmitted. In other words, each PC can transmit at most one data packet per visit of the token.


λ
λ
passage of token
λ
λ
λ
λ

Modelling the system: capturing state

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Since we are not told that they differ in any way we may assume that the four/six PCs have essentially the same behaviour.

However since the order in which the token visits nodes is important we do need to distinguish the PCs.

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In fact for the token ring it is only necessary to represent the location of the token as this is sufficient to determine whether a PC can transmit or not.

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$$\text{Token} \longrightarrow \{1, 2, \ldots, N\}$$

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$$(\underbrace{PC_1}_{\{0,1\}}, \underbrace{PC_2}_{\{0,1\}}, \ldots, \underbrace{PC_N}_{\{0,1\}}, \underbrace{Token}_{\{1,2,\ldots,N\}})$$

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The description of the PC behaviour is very simple in this case. It only has two events which can occur:

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However we will see that we need to be careful and this view will need some refinement.

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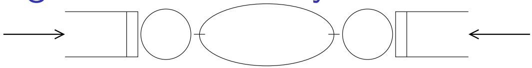
These are two distinct types of event.

Validate against a minimal system



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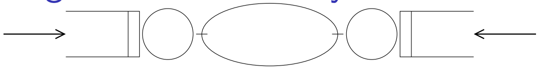

(1,1,1) both PC1 and PC2 ready to transmit; token at PC1

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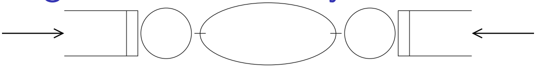

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↓ PC1 transmits

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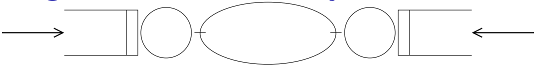

(1,1,1) both PC1 and PC2 ready to transmit; token at PC1

↓ PC1 transmits

(0,1,1) PC2 ready to transmit; token at PC1

↓ token moves to PC2

(0,1,2) PC2 ready to transmit; token at PC2

↓

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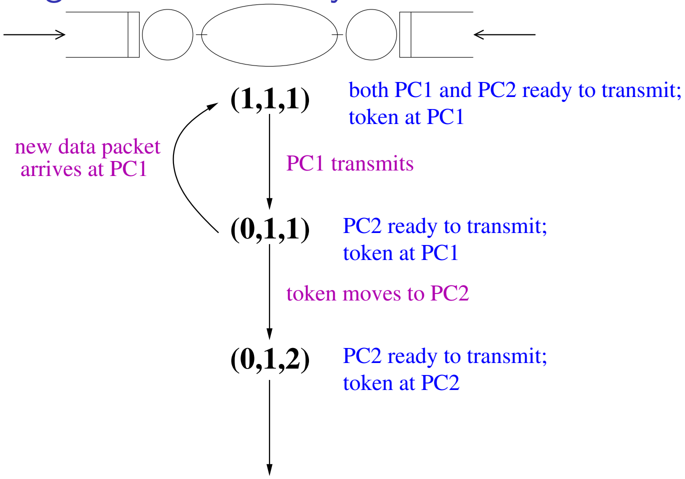

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This will enforce the gated service, i.e. that at most one packet is transmitted per visit of the token.

A refined notion of state

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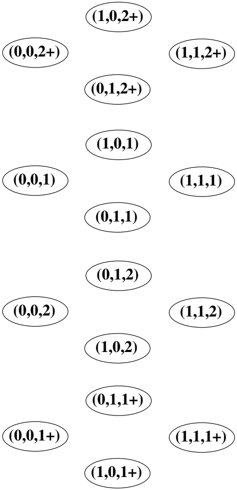
State space for minimal model



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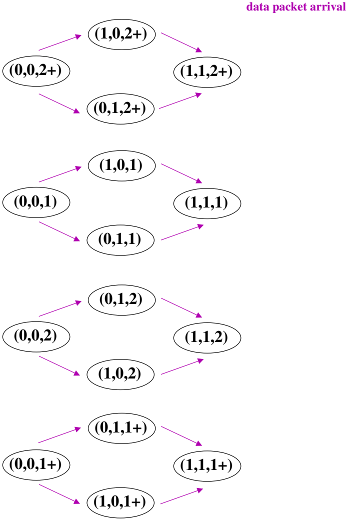

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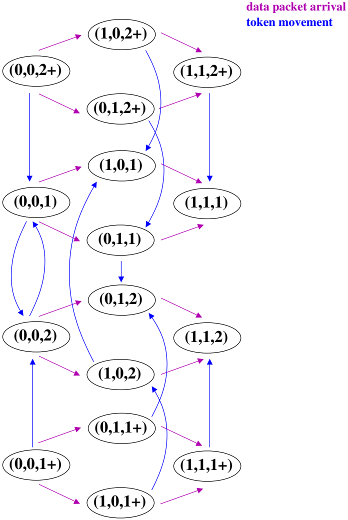

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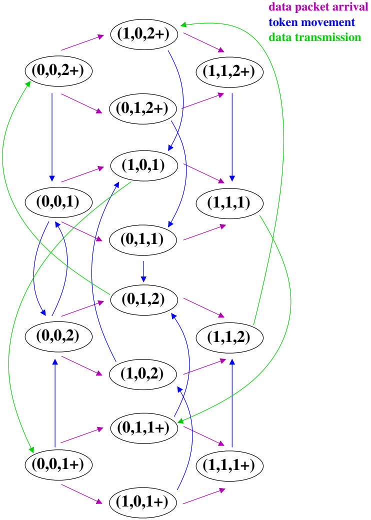

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N	10	20	30
$\|S\|$	2048	4.194304×10^7	6.442450×10^{10}

Performance results

In order to calculate performance measures from the model we must first assign values to the parameters.

The following values were assigned to parameters of the model in both cases (the unit of time is assumed to be milliseconds),

$$\lambda = 0.01 \qquad d = 10 \qquad \mu = 0.1 \qquad m = 1.0 \qquad \omega = 1.0$$

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We wish to calculate the average waiting time of data packets at any PC in the network. Since all the PCs are statistically identical, we can arbitrary choose one as representative—we select PC_1.

Derivation of performance measures

As discussed in the previous lecture there are three different ways in which performance measures can be derived from the steady state distribution of the Markov process, corresponding to different types of measure:

- state-based measures, e.g. utilisation;
- rate-based measures, e.g. throughput;
- other measures which fall outside the above categories, e.g. response time.

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Thus in order to calculate utilisation we sum the steady state probabilities of being in any of the states where the resource is in use.

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Thus to calculate the throughput of the transmission we consider the probability of being a state where tranmission can occur and multiply it by μ the transmission rate.

Calculating waiting time

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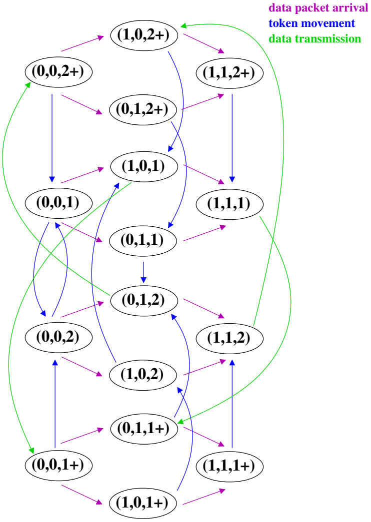
However, if we know the average number of data packets at the interface, and the average throughput, we can apply Little's law to calculate residence time

$$\mathcal{R} = \frac{\mathcal{N}}{X}$$


data packet arrival
token movement
data transmission
(1,0,2+)
(0,0,2+)
(1,1,2+)
(0,1,2+)
(1,0,1)
(0,0,1)
(1,1,1)
(0,1,1)
(0,1,2)
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Similarly, a data transmission can occur whenever there is a data packet waiting and the token is at PC1. So the average throughput X will be the rate at which transmission occurs, μ, multiplied by the total probability of being in one of these states.

Predicted waiting time

Based on these expressions and the calculated steady state distributions we find the following values:
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average waiting time, $\mathcal{W} = \frac{0.1333}{0.008666} - \frac{1}{0.1} = 15.3862 - 10 = 5.3862$

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Thus the average waiting time for data packets will almost double when two more PCs are added to the network.

Difficulties of working with Markov processes

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The solution is to use a high-level modelling language to describe the behaviour we are interested in more closely to the actual observed behaviour of the system, rather than at the level of states and transitions.

High-level modelling formalisms

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We will consider three:

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Each can be used to automatically generate a Markov process and derive performance measures from a high-level description, as we will see.