Performance Modelling — Lecture 1

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Key notions

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Once constructed, such a model becomes a tool with which we can investigate the behaviour of the system.

The discrete event view

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The state of the system is characterised by variables which take distinct values and which change by discrete events, i.e. at a distinct time something happens within the system which results in a change in one or more of the state variables.

The discrete event view: example

We might be interested in the number of nodes in a communication network which are currently waiting to send a message N.

- If a node, which was not previously waiting, generates a message and is now waiting to send then $N \rightarrow N + 1$, or
- If a node, which was previously waiting, successfully transmits its message then $N \rightarrow N - 1$.

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At levels of abstraction above the hardware clock continuous time models are generally appropriate for computer and communication systems.

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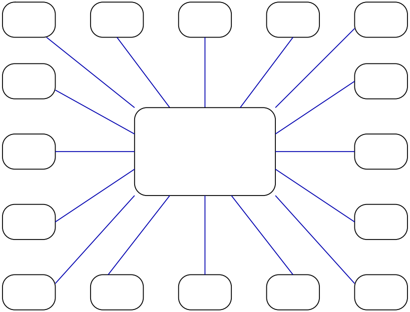
- Users typically want to optimise external measurements of the dynamics such as response time (as small as possible), throughput (as high as possible) or blocking probability (preferably zero);
- In contrast, system managers may seek to optimize internal measurements of the dynamics such as utilisation (reasonably high, but not too high), idle time (as small as possible) or failure rates (as low as possible).

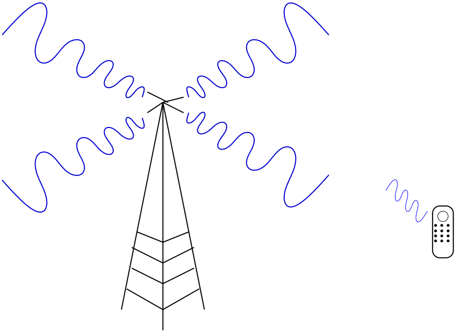
Performance Modelling: Motivation



Capacity planning

- How many clients can the existing server support and maintain reasonable response times?

Performance Modelling: Motivation



Mobile Telephone Antenna

System Configuration

- How many frequencies do you need to keep blocking probabilities low?

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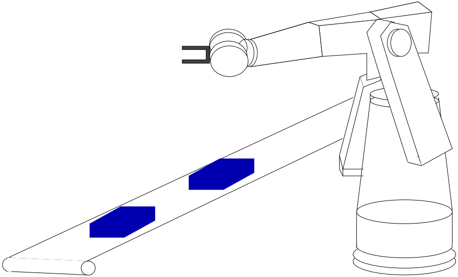
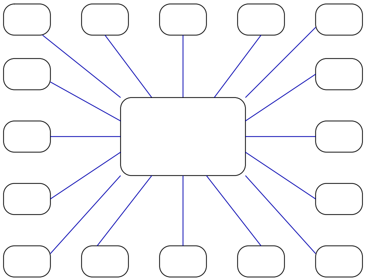


System Tuning

- What speed of conveyor belt will minimize robot idle time and maximize throughput whilst avoiding lost widgets?

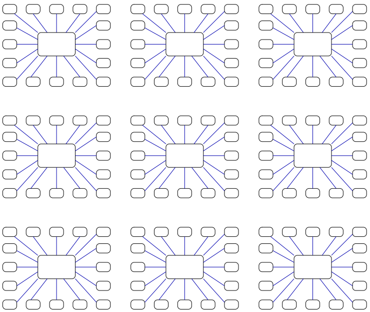
Performance Modelling: Response time analysis



Quality of Service issues

- Can the server maintain reasonable response times?

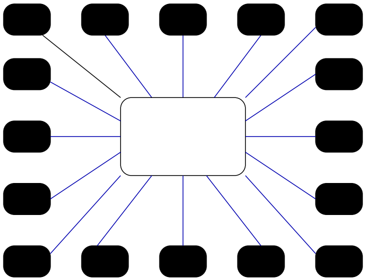
Performance Modelling: Capacity planning



Scalability and capacity planning issues

- How many times do we have to replicate this service to support all of the subscribers?

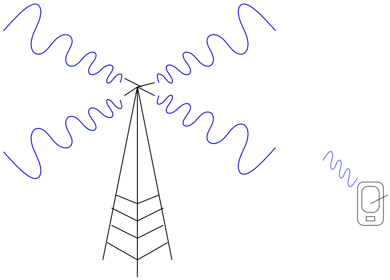
Performance Modelling: Scalability analysis



Robustness and scalability issues

- Will the server withstand a distributed denial of service attack?

Performance Modelling: Service Level Agreements



Service-level agreements

- What percentage of downloads do complete within the time we advertised?

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Probability will be used to represent randomness (e.g. from human users) but also as an abstraction over unknown values (e.g. service times).

Probability Theory

1. There is a sample space, Ω, which encompasses all possible observations or outcomes.
2. There is a collection of subsets of Ω, denoted E, and termed events; these subsets are usually identified as sample points which satisfy some condition.
3. There is a probability mapping, Pr, from E to $\mathbb{R}$. Pr must satisfy three simple conditions:
   1. For any event A, $A \in E$, the mapping Pr is defined and satisfies $0 \leq \Pr(A) \leq 1$.
   2. $\Pr(\Omega) = 1$.
   3. If A and B are mutually exclusive, that is, they contain no sample points in common, then $\Pr(A \cup B) = \Pr(A) + \Pr(B)$.

Simple properties of probabilities

Various properties of probabilities can be derived from the axioms and simple set theory.

For example, the probability of the union of two events A and B which are **not** mutually exclusive is

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B).$$

Similarly, the probability of the complement of event A, denoted by $\neg A$, is

$$\Pr(\neg A) = 1 - \Pr(A).$$

Conditional Probability

The conditional probability of A occurring, given that B has occurred, is

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- If A and B are mutually exclusive $\Pr(A \mid B) = 0$.
- If B is a precondition for A, then $\Pr(A \cap B) = \Pr(A)$.
- Two events are independent if knowledge of the occurrence of one of them tells us nothing about the probability of the other, i.e. $\Pr(A \mid B) = \Pr(A)$, or

$$\boxed{\Pr(A \cap B) = \Pr(A) \times \Pr(B).}$$

Random experiments and events

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- These individual outcomes are also called sample points or elementary events.
- An event is a subset of a sample space.

Random variables

We are interested in the dynamics of a system as events happen over time.

A function which associates a (real-valued) number with the outcome of an experiment is known as a random variable.

Formally, a random variable X is a real-valued function defined on a sample space Ω.

Distribution function

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A random variable X is continuous if $p(x) = 0$ for all real x.

(If X is a continuous random variable, then X can assume infinitely many values, and so it is reasonable that the probability of its assuming any specific value we choose beforehand is zero.)

Mean, or expected value

If X is a continuous random variable with density function $f(\cdot)$, we define the mean or expected value of X, $\mu = E[X]$ by

$$\mu = E[X] = \int_{-\infty}^{\infty} xf(x)dx$$

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If X is a discrete random variable with probability mass function $p(\cdot)$, we define the mean or expected value of $X \in S$, $\mu = E[X]$ by

$$E(X) = \sum_{x \in S} x p(x)$$

Variance

The expectation only gives us an idea of the average value assumed by a random variable, not how much individual values may differ from this average.

The variance, $Var(X)$, gives us an indication of the "spread" of values:

$$\boxed{Var(X) = E[(X - E[X])^2] = E[X^2] - E[X]^2.}$$

Exponential random variables, distribution function

The random variable X is said to be an exponential random variable with parameter λ ($\lambda > 0$) or to have an exponential distribution with parameter λ if it has the distribution function

$$F(x) = \begin{cases} 1 - e^{-\lambda x} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

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Some authors call this distribution the negative exponential distribution.

Exponential random variables, density function

The density function $f = dF/dx$ is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

Mean, or expected value, of the exponential distribution

Suppose X has an exponential distribution with parameter $\lambda > 0$. Then

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Exponential inter-event time distribution

The time interval between successive events can also be deduced.

Let $F(t)$ be the distribution function of T, the time between events. Consider $\Pr(T > t) = 1 - F(t)$:

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Memoryless property of the exponential distribution

The exponential distribution is said to have the memoryless property because the time to the next event is independent of when the last event occurred.

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This value is independent of t (and so the time already spent has not been remembered).

The Poisson distribution

The exponential distribution function is closely related to a discrete random variable, the Poisson distribution.

This random variable takes values in the set $\{0, 1, 2, \ldots\}$ and has the mass function

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The expectation of a Poisson random variable with parameter μ is also μ.

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If we observe a Poisson process with parameter μ for some short time period of length h then:

- the probability that one event occurs is $\mu h + o(h)$.
- the probability that two or more events occur is $o(h)$.
- the probability that no events occur is $1 - \mu h + o(h)$.

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If the occurrence of events is governed by a Poisson distribution then the inter-event times are governed by an exponential distribution with the same parameter, and vice versa.

Therefore, if we know that the delay until an event is exponentially distributed then the probability that it will occur in the infinitesimal time interval of length dt, is μdt.

Superposition and decomposition of exponential distributions

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Then if the inter-event time for any events is exponentially distributed with parameter λ, then the inter-event time for type A events is $p_A \times \lambda$ and similarly for type B events it is $p_B \times \lambda$.

Topics Covered in the Course

- Operational Laws
- Modelling with Continuous Time Markov Chains
- High-level modelling formalisms:
  - Queueing Networks
  - Stochastic Petri Nets
  - Stochastic Process Algebra
- Stochastic Simulation
- Model Parameterisation
- Model Verification and Validation

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- There will also be opportunities to develop your modelling skills through formative exercises in lectures and between lectures.